Generalized Snell's laws for rough interfaces

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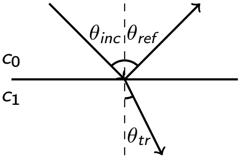
1. Introduction

Introduction

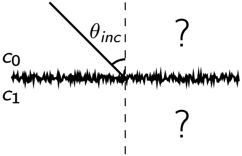
Back in high school, you may have studied the Snell's laws of reflection and refraction for a **flat interface**.



- Reflexion: $\theta_{ref} = \theta_{inc}$.
- Refraction (transmission):

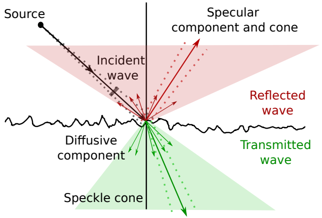
$$\frac{\sin(\theta_{inc})}{c_0} = \frac{\sin(\theta_{tr})}{c_1}.$$

Question: For a rough (random) interface, can we write down similar laws of reflection en refraction?



Introduction

- Wave scattering by rough surfaces is at the heart of several branches of physics and engineering: optics, remote sensing, radar technology, environmental monitoring, communications, non-destructive testing, tissue imaging, laser therapy, etc...
- There is an extensive literature on wave scattering from rough surfaces and interfaces, providing numerous approaches to describe these phenomena.
- The main techniques are gathered in the following review papers:
  - F. G. Bass and I. M. Fuks, Wave scattering from statistically rough surfaces, International series in natural philosophy, Elsevier, 2013;
  - J. A. Ogilvy, Theory of wave scattering from random rough surfaces, CRC Press, 1991.



Introduction

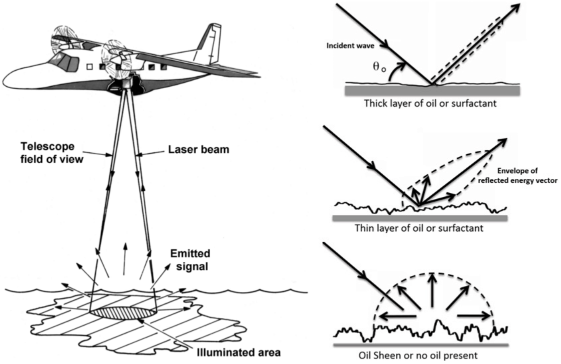


Figure: Oil spill detection with remote sensors, K. Pilzis and V. Vaisis, Environmental Science, Engineering, 2016

Introduction

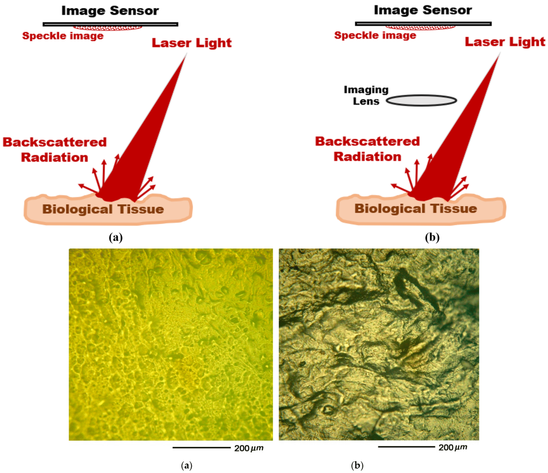


Figure: Nanoscale quantitative surface roughness measurement of articular cartilage using second-order statistical-based biospeckle, D. Youssef, S. H. Elnaby, H. El-Ghandoor, PLoS One, 2021

Introduction

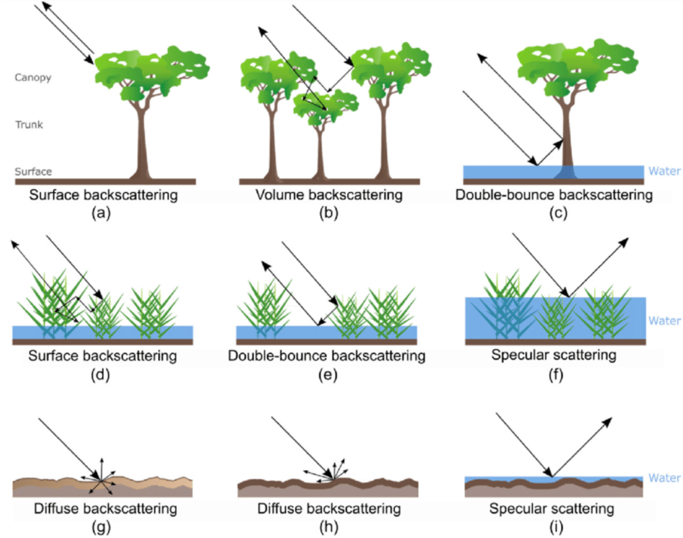


Figure: Spaceborne L-Band Synthetic Aperture Radar Data for Geoscientific Analyses in Coastal Land Applications: A Review, M. Ottinger, C. Kuenzer, Remote sensing, 2020

Generalized Snell's laws can be derived[1][2] for a metasurface.

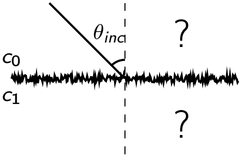
- Reflexion:

$$\sin(\theta_{ref}) - \sin(\theta_{inc}) = \frac{\lambda_0 c_0}{2\pi}\frac{d\Phi}{dx}$$

- Refraction (transmission):

$$\frac{\sin(\theta_{tr})}{c_1} - \frac{\sin(\theta_{inc})}{c_0} = \frac{\lambda_0}{2\pi}\frac{d\Phi}{dx}$$

Here λ_0 the wavelength of the incident wave, and Φ represents a phase discontinuity.
Question: For a rough (random) interface, can we write down similar laws involving the statistical description of the interface?

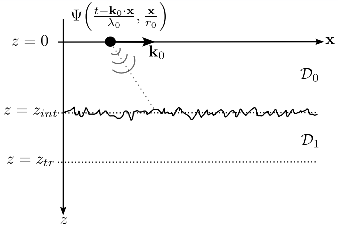


[1] N Yu et al. "Light propagation with phase discontinuities: generalized laws of reflection and refraction". In: *Science* 334 (2011), pp. 333–337.

[2] E. Rousseau and D. Felbacq. "Detailed derivation of the generalized Snell–Descartes laws from Fermat's principle". In: *J. Opt. Soc. Am. A* 40 (2023), pp. 676–681.

2. Physical setting

Physical setting



- A three-dimensional linear wave propagation modeled is considered through the scalar wave equation:

$$\Delta u - \frac{1}{c^2(\mathrm{x}, z)}\partial_{tt}^2 u = \nabla \cdot F(t, \mathrm{x}, z) \qquad (t, \mathrm{x}, z) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R},$$

- equipped with null initial conditions

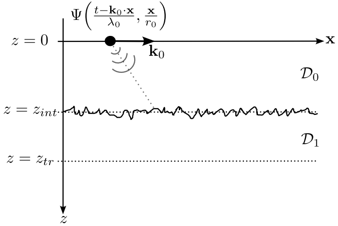
$$u(t = 0, \mathrm{x}, z) = \partial_t u(t = 0, \mathrm{x}, z) = 0 \qquad (\mathrm{x}, z) \in \mathbb{R}^2 \times \mathbb{R},$$

- and continuity conditions at the interface.
- The forcing term

$$F(t, \mathrm{x}, z) := \Psi\left(\frac{t - \mathrm{k}_0 \cdot \mathrm{x}}{\lambda_0}, \frac{\mathrm{x}}{r_0}\right)\delta(z)\mathrm{e}_z,$$

  where $\mathbf{e}_z$ denotes the unit vector pointing in the z-direction.

Physical setting



- The propagation medium consists of two homogeneous subdomains separated by a randomly perturbed interface around $z = z_{int}$:

$$\mathscr{D}_0 := \{(\mathrm{x}, z) \in \mathbb{R}^2 \times \mathbb{R} \quad \text{s.t.} \quad z < z_{int} + \sigma V(\mathrm{x}/\ell_c)\} \tag{1}$$

  and

$$\mathscr{D}_1 := \{(\mathrm{x}, z) \in \mathbb{R}^2 \times \mathbb{R} \quad \text{s.t.} \quad z > z_{int} + \sigma V(\mathrm{x}/\ell_c)\}, \tag{2}$$

  where V is a **mean-zero stationary** random field.
- The velocity field for the wave equation is given by

$$c(\mathrm{x}, z) := \begin{cases} c_0 & \text{if} \quad (\mathrm{x}, z) \in \mathscr{D}_0, \\ c_1 & \text{if} \quad (\mathrm{x}, z) \in \mathscr{D}_1. \end{cases}$$

Physical setting

The mathematical analysis is based on a separation of scales technique. The scales of interest are:

- the wavelength λ_0,
- the typical propagation distance L,
- the beam width r_0,
- the correlation/characteristic scale ℓ_c,
- the amplitude of the fluctuations σ.

We assume:

- $\lambda_0/L \ll 1$ (high-frequency regime);
- $r_0^2/\lambda_0 \sim L$ (paraxial/parabolic scaling);
- $\sigma \sim \lambda_0$ (moderate roughness);
- $\lambda_0 \lesssim \ell_c \lesssim r_0$.

For simplicity, we consider the dimensionless scaling

$$L = 1, \qquad \lambda_0 = \varepsilon, \qquad r_0 = \sqrt{\varepsilon}, \qquad \sigma = \varepsilon, \qquad \text{and} \qquad \ell_c = \varepsilon^\gamma,$$

with

$$\gamma \in [1/2, 1] \qquad \text{and} \qquad \varepsilon \ll 1.$$

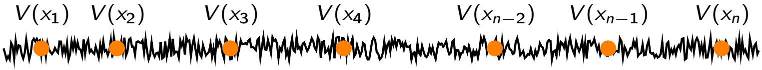
Physical setting

- The random fluctuations are assumed to satisfy ***mixing* properties** describing the loss of statistical dependency for V over the interface.
- Formally, for locations $\mathrm{x}_1, \dots, \mathrm{x}_n \in \mathbb{R}^2$, the random variables

$$V(\mathrm{x}_1), \dots, V(\mathrm{x}_n)$$

tend to be independent as

$$\min_{j,l \in \{1,\dots,n\}} |\mathrm{x}_j - \mathrm{x}_l| \to \infty.$$



Physical setting

More rigorously:

- Introduce

$$\alpha(r) := \sup_{\substack{S,S' \subset \mathbb{R}^2 \\ d(S,S')>r}} \sup_{\substack{A \in \sigma(V(x),\, x \in S) \\ B \in \sigma(V(x),\, x \in S')}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|,$$

  where

$$d(S,S') = \inf_{\substack{s \in S \\ s' \in S'}} |s - s'|.$$

- The value $\alpha(r)$ quantifies the degree of statistical dependency for the random field V over pair of regions at distance at least r.
- The α-mixing property consists in assuming

$$\alpha(r) \to 0 \qquad \text{as} \qquad r \to \infty,$$

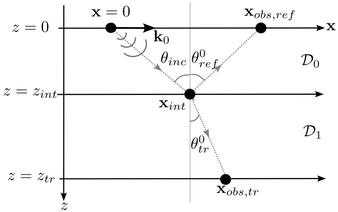
  giving a vanishing of the statistical dependency between $V_{|S}$ and $V_{|S'}$ as the distance between S and S' tends to infinity.

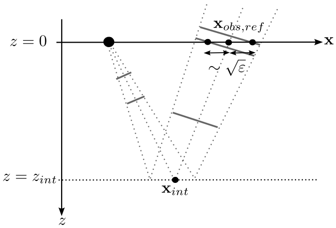
3. Results

3.1 Description of the specular components

For simplicity, we only focus on the reflected component.

Description of the specular components





The specular reflected wave front is given by:

$$U^{ref}(s, \mathrm{y}) := \lim_{\epsilon \to 0} u^{\epsilon, ref}\left(t^{\epsilon}_{obs,ref}(\mathrm{y}) + \epsilon s, \mathrm{x}_{obs,ref} + \sqrt{\epsilon}\,\mathrm{y}, z = 0\right)$$

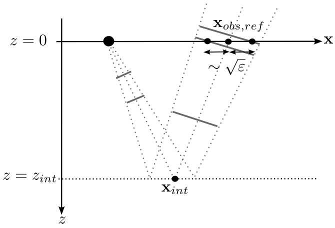
where $t^{\epsilon}_{obs,ref}(\mathrm{y}) = 2z_{int}/(c_0^2 s_0) + \sqrt{\varepsilon}\,\mathrm{k}_0 \cdot \mathrm{y}$ and

$$\mathrm{x}_{obs,ref} = 2\mathrm{k}_0 z_{int}/s_0 = 2\mathrm{x}_{int}$$

with $s_0 := \sqrt{1 - c_0^2 |\mathrm{k}_0|^2}/c_0$.

From the observation points, we can derive the standard Snell's law of reflection and refraction.

Description of the specular components



For a flat interface, the reflected wave front profile is given by

$$U^{ref}(s, \mathrm{y}) = \frac{\mathcal{R}}{2(2\pi)^3} \iint e^{-i\omega(s - \mathrm{q}\cdot\mathrm{y})} \hat{\mathcal{U}}_0(\omega, \mathrm{q}, 2z_{int}) \hat{\Psi}(\omega, \mathrm{q}) \omega^2 d\omega d\mathrm{q},$$

where $\mathcal{R}$ is a reflexion coefficient. Here,

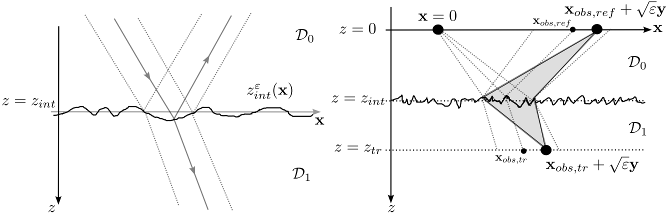
$$\hat{\mathcal{U}}_0(\omega, \mathrm{q}, z) := e^{-i\omega z c_0 \mathrm{q}^T A_0 \mathrm{q}},$$

where A_0 is defined by

$$A_0 := \frac{1}{2c_0^3 \mathfrak{s}_0^3} \left(\mathrm{I}_2 - c_0^2 \, \mathrm{k}_0^\perp (\mathrm{k}_0^\perp)^T\right).$$

$\mathcal{U}_0(\omega, \cdot, \cdot)$ is the **solution to the paraxial wave equation** in homogeneous media.

Description of the specular components



Assuming $\ell_c \sim r_0$ $(\gamma = 1/2)$, the wave front profile reads

$$U^{ref}(s, \mathrm{y}) := \frac{\mathcal{R}}{2(2\pi)^5} \iiiint e^{-i\omega(s - 2\mathfrak{s}_{\mathbf{0}}\mathcal{V}(\mathrm{y}') - \mathbf{q}\cdot\mathrm{y})} e^{i\omega(\mathbf{q}' - \mathbf{q})\cdot\mathrm{y}'}$$
$$\times\, \hat{\mathcal{U}}_0(\omega, \mathbf{q}, z_{int})\hat{\mathcal{U}}_0(\omega, \mathbf{q}', z_{int})\hat{\Psi}(\omega, \mathbf{q}')\omega^4 d\omega d\mathrm{y}' d\mathbf{q}' d\mathbf{q},$$

where $\mathcal{V}$ is a random field with the same law as the original fluctuations V.
In this case, the L^2 of U^{ref} is "conserved", all the energy is carried by the specular component described by the standard Snell's law.

Description of the specular components

Assuming $\lambda_0 \lesssim \ell_c \ll r_0$ ($\gamma \in (1/2, 1]$), the wave front profile reads

$$U^{ref}(s, \mathrm{y}) = \frac{\mathscr{R}}{2(2\pi)^3} \iint e^{-i\omega(s - \mathrm{q}\cdot\mathrm{y})} \hat{\mathcal{U}}_0(\omega, \mathrm{q}, 2z_{int}) \hat{\Psi}(\omega, \mathrm{q}) \phi_V(2\omega\mathscr{s}_0) \omega^2 d\omega d\mathrm{q},$$

where

$$\phi_V(u) = \mathbb{E}\left[e^{iuV(0)}\right]$$

is the characteristic function of the elevation $V(0)$.

- The reflected wave front profile is similar to the one obtained for a flat interface.
- Compared to the previous case $\ell_c \sim r_0$, the scattering operator is here homogenized,

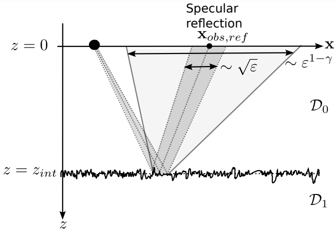
$$\mathbb{E}\left[\int e^{i\omega(\mathrm{q}' - \mathrm{q})\cdot\mathrm{y}'} e^{2i\omega\mathscr{s}_0\mathscr{V}(\mathrm{y}')} d\mathrm{y}'\right] = \delta(\omega(\mathrm{q}' - \mathrm{q}))\phi_V(2\omega\mathscr{s}_0).$$

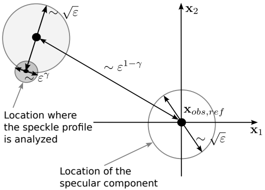
- **The L^2 norm of U^{ref} is no longer "conserved", some energy is missing in the specular component.**

3. Results

3.2 Description of the diffusive (incoherent) components

Description of the diffusive components





The reflected speckle profile refers to the following reflected wavefield

$$S^{\epsilon,ref}_{\bar{s},\bar{y},y}(\widetilde{s},\widetilde{y}) := \epsilon^{-2(\gamma-1/2)} u^{\epsilon,ref}\big(t^{\epsilon}_{obs,ref}(\bar{s},\bar{y},y,\widetilde{y}) + \epsilon\widetilde{s}, x^{\epsilon}_{obs,ref}(\bar{y},y) + \epsilon^{\gamma}\widetilde{y}, z = 0\big),$$

observed at time

$$t^{\epsilon}_{obs,ref}(\bar{s},\bar{y},y,\widetilde{y}) := t_{obs,ref} + \epsilon^{1-\gamma} k_0 \cdot \bar{y} + \sqrt{\epsilon}\, k_0 \cdot y + \epsilon^{\gamma} k_0 \cdot \widetilde{y} + \epsilon^{2(1-\gamma)} \bar{s}.$$

and position

$$x^{\epsilon}_{obs,ref}(\bar{y},y) := x_{obs,ref} + \epsilon^{1-\gamma}\bar{y} + \sqrt{\epsilon}\, y.$$

- The observation point and time exhibit extra terms in $\epsilon^{1-\gamma}$ and $\varepsilon^{2(1-\gamma)}$ corresponding to the roughness parameter $\lambda_0/\ell_c \sim r_0^2/\ell_c$.
- The order of magnitude of the speckle can be understood as spreading a beam width of order r_0 over a two dimensional spatial window of order r_0^2/ℓ_c.

Description of the diffusive components

Proposition

The two-point correlation function of the speckle profile

$$\mathcal{C}^{\epsilon,ref}_{\bar{s},\bar{y},y}(\widetilde{s}_1,\widetilde{y}_1,\widetilde{s}_2,\widetilde{y}_2) := S^{\epsilon,ref}_{\bar{s},\bar{y},y}(\widetilde{s}_1,\widetilde{y}_1)S^{\epsilon,ref}_{\bar{s},\bar{y},y}(\widetilde{s}_2,\widetilde{y}_2).$$

converges in probability in $\mathcal{S}'(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}\times\mathbb{R}^2)$ to

$$\lim_{\epsilon\to 0}\mathcal{C}^{\epsilon,ref}_{\bar{s},\bar{y},y}(\widetilde{s}_1,\widetilde{y}_1,\widetilde{s}_2,\widetilde{y}_2) = C^{ref}_{\bar{s},\bar{y}}(\widetilde{s}_1-\widetilde{s}_2,\widetilde{y}_1-\widetilde{y}_2),$$

where

$$C^{ref}_{\bar{s},\bar{y}}(\widetilde{s},\widetilde{y}) := \frac{\mathcal{R}^2}{4(2\pi)^3}\iint e^{-i\omega(\widetilde{s}-\mathbf{p}\cdot\widetilde{y})}\mathcal{A}(2\mathcal{s}_0,\omega,\mathbf{p})|\hat{\Psi}|_2^2(\omega)\delta(\bar{s}-s^{ref}_{\mathbf{p}})\delta(\bar{y}-y^{ref}_{\mathbf{p}})\omega^2 d\omega d\mathbf{p},$$

with

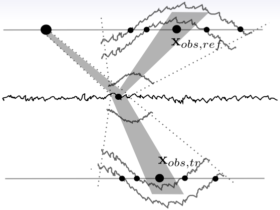
$$\mathcal{A}(v,\omega,\mathbf{p}) := \int\mathbb{E}\big[e^{iv(V(y)-V(0))}\big]e^{-i\omega\mathbf{p}\cdot y}dy,$$

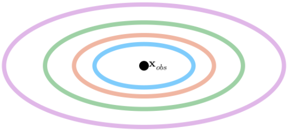
and

$$|\hat{\Psi}|_2^2(\omega) := \frac{\omega^2}{(2\pi)^2}\int|\hat{\Psi}(\omega,q)|^2 dq.$$

We also have,

$$y^{ref}_{\mathbf{p}} := 2z_{int}c_0A_0\mathbf{p} \qquad \text{and} \qquad s^{ref}_{\mathbf{p}} := \mathbf{p}\cdot y^{ref}_{\mathbf{p}}/2 = z_{int}c_0\mathbf{p}^TA_0\mathbf{p}\geq 0.$$





For a given position $\bar{y}$, we have

$$C^{ref}_{\bar{s},\bar{y}}(\tilde{s},\tilde{y}) = \frac{\mathcal{R}^2 c_0^2 s_0^4}{4(2\pi)^3 z_{int}^2}\delta\left(\bar{s} - \frac{\bar{y}^T A_0^{-1}\bar{y}}{2 z_{int} c_0}\right)$$
$$\times \int e^{-i\omega(\tilde{s} - \bar{y}^T A_0^{-1}\tilde{y}/(z_{int}c_0))}|\hat{\Psi}|_2^2(\omega)\mathcal{A}\left(2s_0, \omega, \frac{A_0^{-1}\bar{y}}{z_{int}c_0}\right)\omega^2 d\omega,$$

with

$$A_0^{-1} = c_0 s_0 (I_2 - c_0^2 k_0 \otimes k_0).$$

The observation time to observe the speckle outside the specular cone is given by

$$\bar{s} = \bar{y}^T A_0^{-1}\bar{y}/(2 z_{int} c_0) \geq 0.$$

The temporal duration of the speckle is of order the initial pulse duration characterized by the variable $\tilde{s}$.

Statistics of the diffusive components

- Let us focus on the speckle along the ellipses

$$\hat{\mathcal{S}}^{\epsilon,ref}_{\bar{s},\bar{y},y}(\omega,p) := \hat{S}^{\epsilon,ref}(\bar{s},\bar{y},y,\omega,p)\frac{1}{\epsilon^{3(\gamma-1/2)}}\varphi^{1/2}\Big(2\frac{\bar{s}-s^{ref}_{\mathbf{p}}}{\epsilon^{2(\gamma-1/2)}},2\frac{\bar{y}-y^{ref}_{\mathbf{p}}}{\epsilon^{2(\gamma-1/2)}}\Big),$$

  by windowing of the signal through the function φ.
- This is equivalent to a smoothing of the speckle which is required for technical reasons.

Theorem

For $n \geq 1$ and any fixed $\mathrm{y}_1, \dots, \mathrm{y}_n$, the family $(\hat{\mathcal{S}}^{\epsilon,ref}_{\mathrm{y}_1}, \dots, \hat{\mathcal{S}}^{\epsilon,ref}_{\mathrm{y}_n})_\epsilon$ converges in law to a limit $(\hat{\mathcal{S}}^{ref}_1, \dots, \hat{\mathcal{S}}^{ref}_n)$ made of n independent copies of a complex mean-zero Gaussian random field with covariance function

$$\mathbb{E}[\hat{\mathcal{S}}^{ref}(\bar{s}_1, \bar{\mathrm{y}}_1, \omega_1, \mathrm{p}_1)\hat{\mathcal{S}}^{ref}(\bar{s}_2, \bar{\mathrm{y}}_2, \omega_2, \mathrm{p}_2)] = \hat{\mathcal{K}}_{ref}(\bar{s}_1, \bar{s}_2, \bar{\mathrm{y}}_1, \bar{\mathrm{y}}_2, \omega_1, -\omega_2, \mathrm{p}_1, \mathrm{p}_2)$$

$$\mathbb{E}[\hat{\mathcal{S}}^{ref}(\bar{s}_1, \bar{\mathrm{y}}_1, \omega_1, \mathrm{p}_1)\overline{\hat{\mathcal{S}}^{ref}(\bar{s}_2, \bar{\mathrm{y}}_2, \omega_2, \mathrm{p}_2)}] = \hat{\mathcal{K}}_{ref}(\bar{s}_1, \bar{s}_2, \bar{\mathrm{y}}_1, \bar{\mathrm{y}}_2, \omega_1, \omega_2, \mathrm{p}_1, \mathrm{p}_2)$$

where the kernel $\hat{\mathcal{K}}_{ref}$ is given by

$$\begin{aligned}\hat{\mathcal{K}}_{ref}(\bar{s}_1, \bar{s}_2, \bar{\mathrm{y}}_1, \bar{\mathrm{y}}_2, \omega_1, \omega_2, \mathrm{p}_1, \mathrm{p}_2) = {} & \frac{(2\pi)^3\mathcal{R}^2}{4}\mathcal{A}(2s_0, \omega_1, \mathrm{p}_1)|\hat{\Psi}|_2^2(\omega_1)\hat{\varphi}(\omega_1, \mathrm{p}_1) \\ & \times \delta(\bar{s}_1 - s^{ref}_{\mathrm{p}_1})\delta(\bar{\mathrm{y}}_1 - \mathrm{y}^{ref}_{\mathrm{p}_1})\delta(\bar{s}_1 - \bar{s}_2)\delta(\bar{\mathrm{y}}_1 - \bar{\mathrm{y}}_2) \\ & \times \delta(\omega_1 - \omega_2)\delta(\mathrm{p}_1 - \mathrm{p}_2).\end{aligned}$$

Statistics of the diffusive components

Denote

$$\mathcal{S}_{\mathrm{y}}^{\epsilon,ref}(\bar{s},\bar{\mathrm{y}},\widetilde{s},\widetilde{\mathrm{y}}) := \frac{1}{(2\pi)^3}\iint e^{-i\omega(\widetilde{s}-\mathrm{p}\cdot\widetilde{\mathrm{y}})}\hat{\mathcal{S}}_{\mathrm{y}}^{\epsilon,ref}(\bar{s},\bar{\mathrm{y}},\omega,\mathrm{p})\omega^2 d\omega d\mathrm{p},$$

the inverse Fourier transform of the windowed speckle.

Theorem

For $n \geq 1$ and any fixed $\mathrm{y}_1,\dots,\mathrm{y}_n$, the family $(\mathcal{S}_{\mathrm{y}_1}^{\epsilon,ref},\dots,\mathcal{S}_{\mathrm{y}_n}^{\epsilon,ref})_\epsilon$ converges in law to a limit $(\mathcal{S}_1^{ref},\dots,\mathcal{S}_n^{ref})$ made of n independent copies of a mean-zero Gaussian random field with covariance function

$$\mathbb{E}[\mathcal{S}^{ref}(\bar{s}_1,\bar{\mathrm{y}}_1,\widetilde{s}_1,\widetilde{\mathrm{y}}_1)\mathcal{S}^{ref}(\bar{s}_2,\bar{\mathrm{y}}_2,\widetilde{s}_2,\widetilde{\mathrm{y}}_2)] = \mathcal{K}_{ref}(\bar{s}_1,\bar{s}_2,\bar{\mathrm{y}}_1,\bar{\mathrm{y}}_2,\widetilde{s}_1-\widetilde{s}_2,\widetilde{\mathrm{y}}_1-\widetilde{\mathrm{y}}_2)$$

where the kernel $\mathcal{K}_{ref}$ is given by

$$\begin{aligned}\mathcal{K}_{ref}(\bar{s}_1,\bar{s}_2,\bar{\mathrm{y}}_1,\bar{\mathrm{y}}_2,\widetilde{s},\widetilde{\mathrm{y}}) = \frac{\mathcal{R}^2 c_0^2 \delta_0^4}{4(2\pi)^3 z_{int}^2}\int & e^{-i\omega(\widetilde{s}-\bar{\mathrm{y}}^T A_{\mathbf{0}}^{-\mathbf{1}}\widetilde{\mathrm{y}}/(z_{int}c_{\mathbf{0}}))}|\hat{\Psi}|_2^2(\omega)\\ & \times \mathcal{A}\Big(2\delta_0,\omega,\frac{A_0^{-1}\bar{\mathrm{y}}}{z_{int}c_0}\Big)\hat{\varphi}\Big(\omega,\frac{A_0^{-1}\bar{\mathrm{y}}}{z_{int}c_0}\Big)\omega^2 d\omega\\ & \times \delta\Big(\bar{s}_1-\frac{\bar{\mathrm{y}}_1^T A_0^{-1}\bar{\mathrm{y}}_1}{2z_{int}c_0}\Big)\delta(\bar{s}_1-\bar{s}_2)\delta(\bar{\mathrm{y}}_1-\bar{\mathrm{y}}_2).\end{aligned}$$

Generalized Snell's laws

From the observation point $\mathrm{y}_{\mathrm{p}}^{ref} := 2z_{int}c_0A_0\mathrm{p}$, we derive the generalized Snell's laws for the speckle diffusive components

$$\sin(\theta_{ref}(\mathrm{p})) = \sin(\theta_{inc}) + \mathcal{E}_{ref}(\mathrm{p}) \qquad \text{and} \qquad \frac{\sin(\theta_{tr}(\mathrm{p}))}{c_1} = \frac{\sin(\theta_{inc})}{c_0} + \mathcal{E}_{tr}(\mathrm{p}).$$

For $\lambda_0 \ll \ell_c$ ($\gamma \in (1/2, 1)$), they can be rewritten as

$$\sin(\theta_{ref}(\mathrm{p})) \simeq \sin(\theta_{inc}) + \frac{\lambda_0}{\ell_c}\frac{c_0\mathrm{p}\cdot\hat{k}_0}{\pi} \qquad \text{and} \qquad \frac{\sin(\theta_{tr}(\mathrm{p}))}{c_1} \simeq \frac{\sin(\theta_{inc})}{c_0} + \frac{\lambda_0}{\ell_c}\frac{\mathrm{p}\cdot\hat{k}_0}{\pi}$$

where $\hat{k}_0 = k_0/|k_0|$, and p is distributed according to

$$\mathcal{A}(v,\omega,p) = \int \mathbb{E}[e^{i\omega v(V(y)-V(0))}]e^{i\omega p\cdot y}dy,$$

with

$$v_{ref} = \frac{2\cos(\theta_{inc})}{c_0} \qquad \text{and} \qquad v_{tr} = \frac{\cos(\theta_{inc})}{c_0} - \frac{\cos(\theta_{tr}^0)}{c_1}.$$

This formulation makes the bridge between two approaches in the physical literature[3][4].

[3] A. Santenac and J. Daillant. "Statistical aspects of wave scattering at rough surface". In: *X-ray and Neutron Reflectivity, Lecture Notes in Physics*. Ed. by J. Daillant and A. Gibaud. Vol. 770. Springer, Berlin, Heidelberg, 2009, pp. 59–84.

[4] Yu et al., "Light propagation with phase discontinuities: generalized laws of reflection and refraction".

Generalized Snell's laws

- From the generalized Snell's relations, the reflection and transmission angles can be approximated by

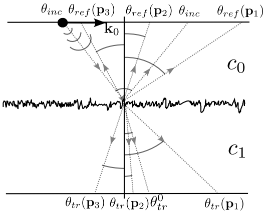
$$\theta_{ref}(\mathrm{p}) \underset{\lambda_0 \ll \ell_c}{\simeq} \theta_{inc} + \frac{\lambda_0}{\pi \ell_c} \frac{\mathrm{p} \cdot \widehat{\mathrm{k}}_0}{\cos(\theta_{inc})} \qquad \text{and} \qquad \theta_{tr}(\mathrm{p}) \underset{\lambda_0 \ll \ell_c}{\simeq} \theta_{tr}^0 + \frac{\lambda_0}{\pi \ell_c} \frac{\mathrm{p} \cdot \widehat{\mathrm{k}}_0}{\cos(\theta_{tr}^0)},$$

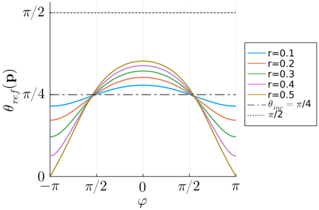
  providing small deviations from the specular refraction and transmission angles of order $\lambda_0/\ell_c \sim r_0^2/\ell_c$.
- For a null incident angle, $\theta_{inc} = 0$, the generalized Snell's relations read

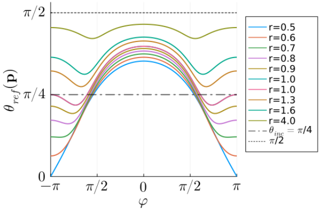
$$\theta_{ref}(\mathrm{p}) = \arctan\left(\frac{\lambda_0 c_0 |\mathrm{p}|}{\pi \ell_c}\right) \qquad \text{and} \qquad \theta_{tr}(\mathrm{p}) = \arctan\left(\frac{\lambda_0 c_1 |\mathrm{p}|}{\pi \ell_c}\right).$$

Generalized Snell's law

For the rough case $\lambda_0 \sim \ell_c$.



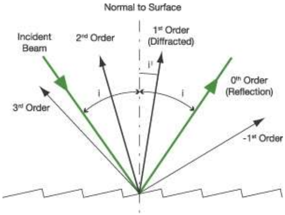


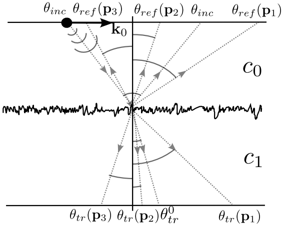


- The corrections in the generalized Snell's law depend on $\mathrm{p} \cdot \widehat{\mathrm{k}}_0$ and $\mathrm{p} \cdot \widehat{\mathrm{k}}_0^\perp$.
- For the numerical illustration we consider

$$\mathrm{p} = \beta_1 \mathrm{k}_0 + \beta_2 \mathrm{k}_0^\perp \qquad \text{with} \qquad (\beta_1, \beta_2) = r(\cos(\varphi), \sin(\varphi)).$$

Comparison with diffraction grating





- Diffraction grating:

$$\sin(\theta_{ref}) = \sin(\theta_{inc}) + m\frac{\lambda_0}{d} \qquad m \in \mathbb{Z},$$

  where d is the distance between successive grooves.

- Scattering by a rough interface

$$\sin(\theta_{ref}(\mathrm{p})) \underset{\lambda_0 \ll \ell_c}{\simeq} \sin(\theta_{inc}) + \frac{c_0 \mathrm{p} \cdot \hat{k}_0}{\pi} \cdot \frac{\lambda_0}{\ell_c}$$

Some ideas of the proofs

From the α-mixing assumption,

$$\alpha(r) \to 0 \qquad \text{as} \qquad r \to \infty,$$

where

$$\alpha(r) := \sup_{\substack{S,S' \subset \mathbb{R}^2 \\ d(S,S') > r}} \sup_{\substack{A \in \sigma(V(x),\, x \in S) \\ B \in \sigma(V(x),\, x \in S')}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|,$$

One can show that for n bounded functions $f_1, \dots, f_n \colon \mathbb{R} \to \mathbb{C}$, and distinct $x_1, \dots, x_n \in \mathbb{R}^2$

$$\lim_{\eta \to 0} \mathbb{E}\Big[\prod_{j=1}^{n} f_j\Big(V\Big(\frac{x_j}{\eta}\Big)\Big)\Big] = \lim_{\eta \to 0} \prod_{j=1}^{n} \mathbb{E}\Big[f_j\Big(V\Big(\frac{x_j}{\eta}\Big)\Big)\Big] = \prod_{j=1}^{n} \mathbb{E}\Big[f_j(V(0))\Big].$$

Some ideas of the proofs

Denoting the wave front

$$U^{\varepsilon,ref}(s,\mathrm{y}) := u^{\epsilon,ref}\left(t^{\epsilon}_{obs,ref}(\mathrm{y}) + \epsilon s, \mathrm{x}_{obs,ref} + \sqrt{\epsilon}\,\mathrm{y}, z=0\right)$$

where $t^{\epsilon}_{obs,ref}(\mathrm{y}) = 2z_{int}/(c_0^2 s_0) + \sqrt{\varepsilon}\,\mathrm{k}_0\cdot\mathrm{y}$ and

$$\mathrm{x}_{obs,ref} = 2\mathrm{k}_0 z_{int}/s_0 = 2\mathrm{x}_{int}.$$

In the case $\ell_c \ll r_0$ ($\gamma > 1/2$), we have

$$\mathbb{E}[U^{\epsilon,ref}(s_1,\mathrm{y}_1,\widetilde{\mathrm{y}}_1)U^{\epsilon,ref}(s_2,\mathrm{y}_2,\widetilde{\mathrm{y}}_2)]$$
$$= \frac{\mathcal{R}^2}{4(2\pi)^{10}}\int\cdots\int e^{-i\omega_1(s_1-\mathrm{q}_1\cdot\mathrm{y}_1)}e^{-i\omega_2(s_2-\mathrm{q}_2\cdot\mathrm{y}_2)}$$
$$\times\,\mathbb{E}\left[e^{2i\omega_1 s_0 V(\mathrm{x}_{int}/\epsilon^{\gamma}+\mathrm{y}'_1/\epsilon^{\gamma-1/2}+\widetilde{\mathrm{y}}_1)}e^{2i\omega_2 s_0 V(\mathrm{x}_{int}/\epsilon^{\gamma}+\mathrm{y}'_2/\epsilon^{\gamma-1/2}+\widetilde{\mathrm{y}}_2)}\right]$$
$$\ldots d\omega_1 d\omega_2 d\mathrm{y}'_1 d\mathrm{y}'_2 d\mathrm{q}'_1 d\mathrm{q}'_2 d\mathrm{q}_1 d\mathrm{q}_2.$$

Some ideas of the proofs

Denoting the wave front

$$U^{\varepsilon,ref}(s,\mathrm{y}) := u^{\epsilon,ref}\left(t^{\epsilon}_{obs,ref}(\mathrm{y}) + \epsilon s, \mathrm{x}_{obs,ref} + \sqrt{\epsilon}\,\mathrm{y}, z=0\right)$$

where $t^{\epsilon}_{obs,ref}(\mathrm{y}) = 2z_{int}/(c_0^2 s_0) + \sqrt{\varepsilon}\,\mathrm{k}_0 \cdot \mathrm{y}$ and

$$\mathrm{x}_{obs,ref} = 2\mathrm{k}_0 z_{int}/s_0 = 2\mathrm{x}_{int}.$$

In the case $\ell_c \ll r_0$ ($\gamma > 1/2$), we have

$$\mathbb{E}[U^{\epsilon,ref}(s_1,\mathrm{y}_1,\widetilde{\mathrm{y}}_1)U^{\epsilon,ref}(s_2,\mathrm{y}_2,\widetilde{\mathrm{y}}_2)]$$

$$= \frac{\mathcal{R}^2}{4(2\pi)^{10}}\int\cdots\int e^{-i\omega_1(s_1-\mathrm{q}_1\cdot\mathrm{y}_1)}e^{-i\omega_2(s_2-\mathrm{q}_2\cdot\mathrm{y}_2)}$$

$$\times\,\mathbb{E}\left[e^{2i\omega_1 s_0 V(\mathrm{x}_{int}/\epsilon^{\gamma}+\mathrm{y}_1'/\epsilon^{\gamma-1/2}+\widetilde{\mathrm{y}}_1)}\right]\mathbb{E}\left[e^{2i\omega_2 s_0 V(\mathrm{x}_{int}/\epsilon^{\gamma}+\mathrm{y}_2'/\epsilon^{\gamma-1/2}+\widetilde{\mathrm{y}}_2)}\right]$$

$$\ldots d\omega_1 d\omega_2 d\mathrm{y}_1' d\mathrm{y}_2' d\mathrm{q}_1' d\mathrm{q}_2' d\mathrm{q}_1 d\mathrm{q}_2.$$

$$\simeq \mathbb{E}[U^{\epsilon,ref}(s_1,\mathrm{y}_1,\widetilde{\mathrm{y}}_1)]\mathbb{E}[U^{\epsilon,ref}(s_2,\mathrm{y}_2,\widetilde{\mathrm{y}}_2)]$$

Some ideas of the proofs

One can also show that for n bounded functions $g_1, \dots, g_n$: $\mathbb{R}^2 \to \mathbb{C}$, $\mathrm{y}_1, \dots, \mathrm{y}_n \in \mathbb{R}^2$, and distinct $\mathrm{x}_1, \dots, \mathrm{x}_n \in \mathbb{R}^2$,

$$\lim_{\eta\to 0} \mathbb{E}\Big[\prod_{j=1}^n g_j\Big(V\Big(\frac{\mathrm{x}_j}{\eta}+\frac{\mathrm{y}_j}{2}\Big), V\Big(\frac{\mathrm{x}_j}{\eta}-\frac{\mathrm{y}_j}{2}\Big)\Big)\Big] = \lim_{\eta\to 0}\prod_{j=1}^n \mathbb{E}\Big[g_j\Big(V\Big(\frac{\mathrm{x}_j}{\eta}+\frac{\mathrm{y}_j}{2}\Big), V\Big(\frac{\mathrm{x}_j}{\eta}-\frac{\mathrm{y}_j}{2}\Big)\Big)\Big]$$

$$= \prod_{j=1}^n \mathbb{E}\Big[g_j\Big(V\Big(\frac{\mathrm{y}_j}{2}\Big), V\Big(-\frac{\mathrm{y}_j}{2}\Big)\Big)\Big].$$

In the case $\ell_c \ll r_0$ ($\gamma > 1/2$), we have

$$\lim_{\epsilon\to 0} \mathbb{E}\big[\big\langle \hat{\mathcal{S}}_{\mathrm{y}}^{\epsilon,ref}, \phi\big\rangle_{\mathcal{S}',\mathcal{S}}^{2n}\big] = \frac{n!}{2^n}\sigma_{ref}^{2n} = \mathbb{E}[\big\langle \hat{\mathcal{S}}_{\mathrm{y}}^{ref}, \phi\big\rangle_{\mathcal{S}',\mathcal{S}}^{2n}].$$

and

$$\lim_{\epsilon\to 0} \mathbb{E}\big[\big\langle \hat{\mathcal{S}}_{\mathrm{y}}^{\epsilon,ref}, \phi\big\rangle_{\mathcal{S}',\mathcal{S}}^{2n+1}\big] = 0.$$

Thank you!