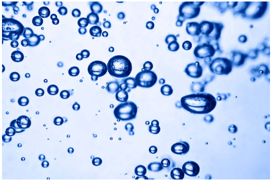
Wave localisation at subwavelength scales

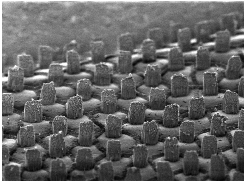
Habib Ammari

Department of Mathematics, ETH Zürich

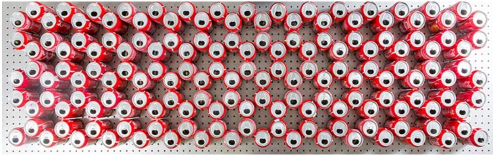
Wave interactions with subwavelength structures



Hear the sound of bubbles



Hair cells in the cochlea



Resonance in air at 80 cm; diameter 6.5 cm

- Subwavelength resonances $\Leftarrow$ High-contrast regime + Long-range interactions.
- Discrete approximations; Capacitance matrix formulation;
- Wave localisation and transport at subwavelength scales: wireless communications; biomedical superresolution imaging; quantum computing.

Mathematical models

- Reciprocal and non-reciprocal transport and wave localisation.
- **Mathematical foundations**: topological interface modes; Anderson localisation; time-modulated systems, non-Hermitian skin effect; localisation in disordered systems.
- Silvio Barandun (ETH), Jinghao Cao (Caltech), **Bryn Davies** (Warwick), Brian Fitzpatrick, Xin Fu (Tsinghua), David Gontier (ENS, Paris), **Erik Hiltunen** (Univ. Oslo), Wenjia Jing (Tsinghua), Thea Kosche (ETH), Hyundae Lee (Inha Univ.), Ping Liu (Zhejiang Univ), Liora Rueff (ETH), Sanghyeon Yu (Korea Univ.), Hai Zhang (UST, Hong Kong), Alexander Uhlmann (ETH).
- **Mathematical theories for metamaterials: From condensed matter theory to subwavelength physics.** NSF-CBMS Regional Conf. Ser., AMS 2025.

Subwavelength resonance problem

- $D_1, D_2, \ldots, D_N \subset \mathbb{R}^d$, $d \in \{2,3\}$, $N \in \mathbb{N}$: disjoint, connected bounded sets with boundaries in $C^{1,s}$ for some $0 < s < 1$; $D = \cup_{i=1}^N D_i$.
- v_i: wave speed in resonator D_i; $k_i = \omega / v_i$: wave number in D_i, where $\omega \in \mathbb{R}, \omega \neq 0,$: operating frequency; v and $k = \omega / v$: wave speed and wave number in the background medium.
- Scattering problem:

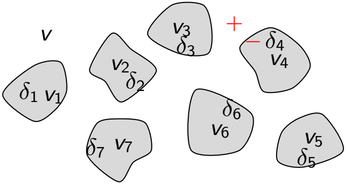
$$
\begin{cases}
\Delta u + k^2 u = 0 & \text{in } \mathbb{R}^d \setminus \overline{D}, \\
\Delta u + k_i^2 u = 0 & \text{in } D_i, \text{ for } i = 1, \ldots, N, \\
u|_+ - u|_- = 0 & \text{on } \partial D, \\
\delta_i \left. \dfrac{\partial u}{\partial \nu} \right|_+ - \left. \dfrac{\partial u}{\partial \nu} \right|_- = 0 & \text{on } \partial D_i \text{ for } i = 1, \ldots, N, \\
u - u_{\mathrm{in}} \text{ satisfies an outgoing radiation condition.}
\end{cases}
$$

- High contrast regime $0 < \delta \ll 1$:

$$
v, v_i = \mathcal{O}(1), \delta_i = \mathcal{O}(\delta), \qquad \text{for } i = 1, \ldots, N.
$$

Subwavelength resonance problem

- Finite collection of resonators:



- Subwavelength resonant frequency: Given $\delta > 0$, a subwavelength resonant frequency $\omega = \omega(\delta) \in \mathbb{C}$:
  (i) there exists a non-trivial solution to the scattering problem with $u_{\text{in}} \equiv 0$, known as an associated resonant mode;
  (ii) ω depends continuously on δ and satisfies $\omega \to 0$ as $\delta \to 0$.

Capacitance formulation of the resonance problem

- For sufficiently small $\delta > 0$, there exist N subwavelength resonant frequencies $\omega_1(\delta), \ldots, \omega_N(\delta)$ with non-negative real parts.
- Generalised capacitance matrix:

$$\mathcal{C} = VC, \qquad V = \begin{pmatrix} \frac{v_1^2\delta_1}{|D_1|} & & \\ & \ddots & \\ & & \frac{v_N^2\delta_N}{|D_N|} \end{pmatrix}$$

- Capacitance matrix of D: $C_{ij} := \int_{\mathbb{R}^3\setminus\overline{D}} \nabla V_i \cdot \nabla V_j \,\mathrm{d}x = -\int_{\partial D_i} \left.\frac{\partial V_j}{\partial \nu}\right|_+ \mathrm{d}\sigma.$

$$\begin{cases} \Delta V_i = 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ V_i = \delta_{ij} & \text{on } \partial D_j, \\ V_i(x) = \mathcal{O}\left(|x|^{-1}\right) & \text{as } |x| \to \infty; \end{cases}$$

Capacitance matrix of a finite system

- C: symmetric; positive definite;
- $C_{ij} < 0$ for any $1 \leq i \neq j \leq N$;
- C strictly diagonally dominant:
$$C_{ii} > \sum_{j \neq i} |C_{ij}|, \text{ for any } 1 \leq i \leq N;$$
- C: nonsingular Minkowski-matrix $\Rightarrow C^{-1}$: Minkowski-matrix; principle minors of C: positive.
- Dilute expansion: $D_i = \epsilon B_i + z_i, \epsilon \to 0$:
$$C_{ii} = \epsilon \mathrm{Cap}_{B_i} + \mathcal{O}(\epsilon^3),$$
$$C_{ij} = -\frac{\epsilon^2 \mathrm{Cap}_{B_i} \mathrm{Cap}_{B_j}}{4\pi |z_i - z_j|} + \mathcal{O}(\epsilon^3), \quad \text{for } i \neq j;$$
- Decay property for N large enough:
$$|C_{ij}| \lesssim \frac{1}{\mathrm{dist}(D_i, D_j)}.$$
- $\Leftarrow C_{ij}^{(N)} \leq C_{ij}^{(N+1)}$; For $i = j \Rightarrow$ diagonal capacitance coefficients increase when adding additional resonators.

Capacitance formulation of the resonance problem

- As $\delta \to 0$,

$$\omega_n = \sqrt{\lambda_n} - \mathrm{i}\tau_n + \mathcal{O}(\delta^{3/2}), \quad n = 1, \dots, N;$$

- λ_n for $n = 1, \dots, N$: eigenvalues of $\mathcal{C}$;
- For each $n = 1, \dots, N$, $\sqrt{\lambda_n} = \mathcal{O}(\delta^{1/2})$ and $\tau_n = \mathcal{O}(\delta)$ as $\delta \to 0$;
- $\mathbf{v}_n$: eigenvector of $\mathcal{C}$ associated to $\lambda_n \Rightarrow$ resonant mode u_n associated to ω_n.
- τ_n ($v_1 = v_2 = \dots = v_N$ and $\delta_1 = \delta_2 = \dots = \delta_N$):

$$\tau_n = \delta_1 \frac{v_1^2}{8\pi v} \frac{\mathbf{v}_n^\top CJC\mathbf{v}_n}{\|\mathbf{v}_n\|_D^2};$$

- J: $N \times N$ matrix of ones; $\mathbf{v}_n$: eigenvector associated to λ_n; $\|x\|_D := \left(\sum_{i=1}^N |D_i| x_i^2\right)^{1/2}$ for $x \in \mathbb{R}^N$.
- V complex $\Rightarrow \mathcal{C}$: may not be diagonalisable $\Leftrightarrow$ exceptional point degeneracy may occur.

Capacitance formulation of the resonance problem

- Single resonator:

$$\omega_1 = \underbrace{\sqrt{\frac{\mathrm{Cap}_D}{|D|}}v_1\sqrt{\delta}}_{:=\omega_M} - \mathrm{i}\underbrace{\left(\frac{\mathrm{Cap}_D^2 v_1^2}{8\pi v|D|}\delta\right)}_{:=\tau_M} + \mathcal{O}(\delta^{\frac{3}{2}});$$

- Monopole approximation:

$$u^s(x) := (u - u_{\mathrm{in}})(x) = g(\omega, \delta, D)(1 + o(1))u_{\mathrm{in}}(0)G^k(x); \quad 0 \in D;$$

- Scattering coefficient $\Rightarrow$ Scattering enhancement near ω_M:

$$g(\omega, \delta, D) = \frac{\mathrm{Cap}_D}{1 - (\frac{\omega_M}{\omega})^2 + \mathrm{i}\gamma_M};$$

- Damping constant:

$$\gamma_M := \frac{\omega(v + v_1)\mathrm{Cap}_D}{8\pi v v_1} - \frac{(v - v_1)}{v}\frac{\delta v_1 \mathrm{Cap}_D^2}{8\pi|D|\omega}.$$

Capacitance formulation of the resonance problem

- Parity-symmetric dimer (with respect to 0):

$$\omega_1 = \underbrace{\sqrt{(C_{11} + C_{12})} v_1 \sqrt{\delta}}_{:=\omega_{M,1}} - \mathrm{i}\tau_1 \delta + \mathcal{O}(\delta^{3/2});$$

$$\omega_2 = \underbrace{\sqrt{(C_{11} - C_{12})} v_1 \sqrt{\delta}}_{:=\omega_{M,2}} + \delta^{3/2}\widehat{\eta}_1 + \mathrm{i}\delta^2 \widehat{\eta}_2 + \mathcal{O}(\delta^{5/2});$$

- $\widehat{\eta}_1, \widehat{\eta}_2$: real numbers determined by D, v, and v_1;

$$\tau_1 = \frac{v_1^2}{4\pi v}(C_{11} + C_{12})^2.$$

- ω_1 and ω_2: monopole and dipole hybridised resonances of the resonator dimer D.
- $\omega_{M,1} < \omega_{M,2}$; $\Im \omega_1 = \mathcal{O}(\delta)$ while $\Im \omega_2 = \mathcal{O}(\delta^2)$.
- Point scatterer with resonant monopole and resonant dipole modes:

$$\underbrace{g^0(\omega) u_{\mathrm{in}}(0) G^k(x)}_{\text{monopole}} + \underbrace{\nabla u_{\mathrm{in}}(0) \cdot g^1(\omega) \nabla G^k(x)}_{\text{dipole}}.$$

Capacitance formulation for nonlinear systems

- Nonlinear model:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^d \setminus \overline{D}, \\ \Delta u + k_i^2 u + \beta_i g(\omega) f(u) = 0 & \text{in } D_i, \text{ for } i = 1, \ldots, N, \\ u|_+ - u|_- = 0 & \text{on } \partial D, \\ \delta_i \dfrac{\partial u}{\partial \nu}\Big|_+ - \dfrac{\partial u}{\partial \nu}\Big|_- = 0 & \text{on } \partial D_i \text{ for } i = 1, \ldots, N, \\ u \text{ satisfies an outgoing radiation condition.} \end{cases}$$

- Kerr-type nonlinearity: $g(\omega) = |\omega|^2 \omega$; $f(u) = |u|^2 u$.
- Saturable nonlinearity $f(u) = \dfrac{|u|^2}{(1 + |u|^2)} u$.
- One-to-one correspondence between the subwavelength resonant frequencies and the solutions to

$$(\mathcal{C} - \omega^2) \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{pmatrix} + g(\omega) V^2 \begin{pmatrix} \beta_1 f(q_1) \\ \beta_2 f(q_2) \\ \vdots \\ \beta_N f(q_N) \end{pmatrix} = 0; \quad V_{ij} := \frac{1}{|B_j|} \delta_{ij};$$

Capacitance formulation for nonlinear systems

- Nonlinearity $-\beta\omega^2|u|^2 u$.
- $\mathcal{C}q - \omega^2\left(q - \beta V^2|q|^2 q\right) = 0$.
- $D = B_1 \cup B_2$: preserved under the mirror symmetry $P(x_1, x_2, x_3) = (x_1, x_2, -x_3)$ and $P(B_1) = P(B_2) \Rightarrow$ for all $a \in \mathbb{C}$,

$$v_0 = \begin{pmatrix} a \\ a \end{pmatrix} \text{ and } v_1 = \begin{pmatrix} -a \\ a \end{pmatrix}$$

with

$$\omega_0 = \frac{\mathcal{C}_{11} + \mathcal{C}_{22}}{\left(1 - \beta\frac{2}{|B_1|^2}|a|^2\right)} \quad \text{and} \quad \omega_1 = \frac{\mathcal{C}_{11} - \mathcal{C}_{22}}{\left(1 - \beta\frac{2}{|B_1|^2}|a|^2\right)}.$$

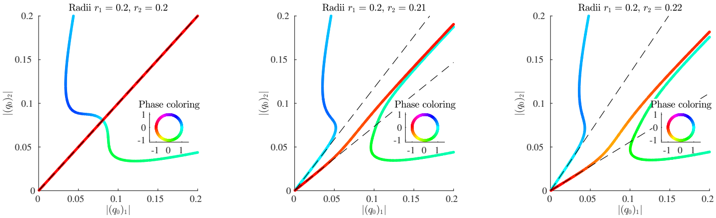
- If (ω_0, q_0): solution $\Rightarrow$

$$(\omega_0, p_0) \quad \text{with} \quad p_0 = \begin{pmatrix} (q_0)_2 \\ (q_0)_1 \end{pmatrix}$$

solution and the phase of $(p_0)_1/(p_0)_2$: conjugate to the phase of $(q_0)_1/(q_0)_2$.

Capacitance formulation for nonlinear systems

- Nonlinearity-induced subwavelength resonant frequencies:



Capacitance formulation for time-modulated systems

- Time-modulation of the resonators:

$$\kappa(x,t) = \begin{cases} \kappa, & x \in \mathbb{R}^d \setminus \overline{D}, \\ \kappa_r \kappa_i(t), & x \in D_i, \end{cases}, \qquad \rho(x,t) = \begin{cases} \rho, & x \in \mathbb{R}^d \setminus \overline{D}, \\ \rho_r \rho_i(t), & x \in D_i. \end{cases}$$

- $\rho_i(t), \kappa_i(t)$: modulation inside D_i; ρ_i, κ_i: periodic with period T;
- Find $\omega \in Y_t^* := \mathbb{C}/(\Omega\mathbb{Z}$; $\Omega = (2\pi)/T = \mathcal{O}(\delta^{1/2})$.

$$\begin{cases} \left(\dfrac{\partial}{\partial t} \dfrac{1}{\kappa(x,t)} \dfrac{\partial}{\partial t} - \nabla \cdot \dfrac{1}{\rho(x,t)} \nabla \right) u(x,t) = 0, \\ u(x,t)e^{-i\omega t} \text{ is } T\text{-periodic in } t. \end{cases}$$

- A quasifrequency is a subwavelength quasifrequency if the corresponding solution is essentially supported in the subwavelength frequency regime.

Capacitance formulation for time-modulated systems

- Capacitance matrix formulation of the problem:
  - As $\delta \to 0$, the quasifrequencies $\omega \in Y_t^*$ are, to leading order, given by the quasifrequencies of the system of ordinary differential equations:

$$\sum_{j=1}^{N} C_{ij} y_j(t) = -|D_i| \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{\delta_i v_i^2} \frac{\mathrm{d}y_i}{\mathrm{d}t} \right),$$

    for $i = 1, \ldots, N$. $(y_j(t) = e^{i\omega t} \sum_n y_{j,n} e^{in\Omega t})$.
  - Rewrite as a system of Hill equations:

$$\Psi''(t) + M(t)\Psi(t) = 0.$$

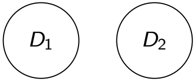
  - Compute the Floquet exponents of the Hill system of equations.
  - If $\delta_i v_i^2$ independent of t for $i = 1, \ldots, N$ $(= \delta v_r^2)$:

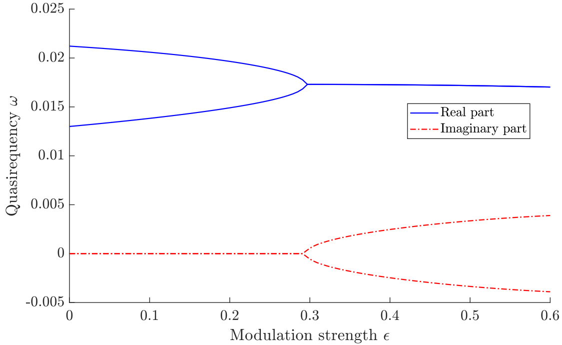
$$\Psi''(t) + \mathcal{C}\Psi(t) = 0.$$

  - $\Rightarrow$ Static case: Quasifrequencies $\omega_i = \sqrt{\lambda_i}$ at leading order in δ.

Capacitance formulation for time-modulated systems

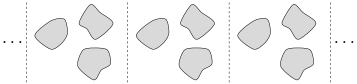
- The two subwavelength quasifrequencies of a pair of $\mathcal{PT}$-symmetric, time-modulated resonators;
- $\kappa_1(t) = 1 + \epsilon \sin(\Omega t)$; $\kappa_2(t) = 1 - \epsilon \sin(\Omega t)$;
- Exceptional point in the time-modulated case: $\epsilon \approx 0.3$.





Capacitance formulation for infinite periodic systems

- d_l: dimension of periodicity of the lattice. d: dimension of the ambient space.
- Three different cases:

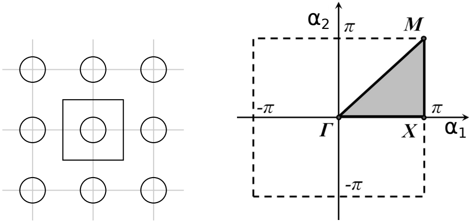


  - $d - d_l = 0$: crystal;
  - $d - d_l = 1$: screen;
  - $d - d_l = 2$: chain.
- Λ: periodic lattice; Y: fundamental domain; Λ^*: dual lattice of Λ; Brillouin zone $Y^* := \left(\mathbb{R}^{d_l} \times \{\mathbf{0}\}\right)/\Lambda^*$; $\mathbf{0}$: zero-vector in $\mathbb{R}^{d-d_l}$; $x = (x_l, x_0)$.
  - $P_l : \mathbb{R}^d \to \mathbb{R}^{d_l}$: projection onto the first d_l coordinates; $P_\perp : \mathbb{R}^d \to \mathbb{R}^{d-d_l}$: projection onto the last $d - d_l$ coordinates.
  - $l_1, \ldots, l_{d_l} \in \mathbb{R}^d$: lattice vectors generating the lattice Λ; $P_\perp l_i = 0$; $\Lambda := \left\{ m_1 l_1 + \cdots + m_{d_l} l_{d_l} \mid m_i \in \mathbb{Z} \right\}$.
  - Λ^*: generated by $\alpha_1, \ldots, \alpha_{d_l}$ s.t. $\alpha_i \cdot l_j = 2\pi\delta_{ij}$ and $P_\perp \alpha_i = 0$.
  - $Y := \left\{ c_1 l_1 + \cdots + c_{d_l} l_{d_l} \mid 0 \leq c_1, \ldots, c_{d_l} \leq 1 \right\}$.
- Periodically repeated i^{th} $\mathcal{D}_i$ and the full periodic structure $\mathcal{D}$:

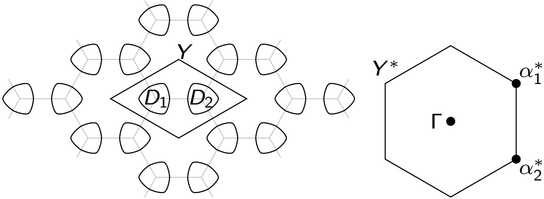
$$\mathcal{D}_i = \bigcup_{m \in \Lambda} D_i + m, \qquad \mathcal{D} = \bigcup_{i=1}^{N} \mathcal{D}_i.$$

Capacitance formulation for infinite periodic systems

- Square lattice and corresponding Brillouin zone:



- Honeycomb lattice and corresponding Brillouin zone:



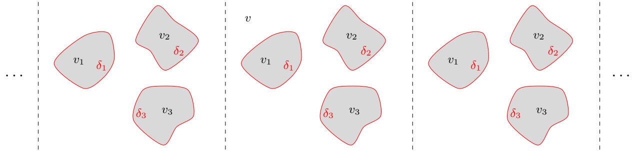
Capacitance formulation for infinite periodic systems

- Infinite periodic structure:

$$D_i^m = D_i + m, \qquad \mathcal{D}_i = \bigcup_{m\in\Lambda} D_i + m, \qquad \mathcal{D} = \bigcup_{i=1}^{N} \mathcal{D}_i .$$

- Resonance problem:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^d \setminus \mathcal{D}, \\ \Delta u + k_i^2 u = 0 & \text{in } \mathcal{D}_i, \ i = 1,\ldots,N, \\ u|_+ - u|_- = 0 & \text{on } \partial\mathcal{D}, \\ \delta_i \dfrac{\partial u}{\partial \nu}\Big|_+ - \dfrac{\partial u}{\partial \nu}\Big|_- = 0 & \text{on } \partial\mathcal{D}_i, \ i = 1,\ldots,N, \\ u(x_l, x_0) & \text{satisfies the outgoing radiation condition as } |x_0| \to \infty. \end{cases}$$



Capacitance formulation for infinite periodic systems

- $f(x) \in L^2(\mathbb{R}^d)$: α-quasiperiodic, with quasiperiodicity $\alpha \in Y^*$, if $e^{-\mathrm{i}\alpha\cdot x} f(x)$: Λ-periodic;
- Floquet transform of $f \in L^2(\mathbb{R}^d)$:

$$\mathcal{U}[f](x,\alpha) := \sum_{m\in\Lambda} f(x-m)e^{\mathrm{i}\alpha\cdot m}, \quad x,\alpha \in \mathbb{R}^d.$$

- $\mathcal{U}[f]$: α-quasiperiodic in x and periodic in α.
- Floquet transform: invertible map $\mathcal{U} : L^2(\mathbb{R}^d) \to L^2(Y \times Y^*)$, with inverse given by

$$\mathcal{U}^{-1}[g](x) = \frac{1}{|Y_l^*|} \int_{Y^*} g(x,\alpha)\,\mathrm{d}\alpha, \quad x \in \mathbb{R}^d,$$

$g(x,\alpha)$: extended quasiperiodically for x outside of the unit cell Y.

Capacitance formulation for infinite periodic systems

- $u^\alpha(x) := \mathcal{U}[u](x,\alpha)$:

$$\begin{cases} \Delta u^\alpha + k^2 u^\alpha = 0 \quad \text{in } \mathbb{R}^d \setminus \mathcal{D}, \\ \Delta u^\alpha + k_i^2 u^\alpha = 0 \quad \text{in } \mathcal{D}_i, \ i = 1, \ldots, N, \\ u^\alpha|_+ - u^\alpha|_- = 0 \quad \text{on } \partial\mathcal{D}, \\ \delta_i \dfrac{\partial u^\alpha}{\partial \nu}\Big|_+ - \dfrac{\partial u^\alpha}{\partial \nu}\Big|_- = 0 \quad \text{on } \partial\mathcal{D}_i, \ i = 1, \ldots, N, \\ u^\alpha(x_l, x_0) \text{ is } \alpha\text{-quasiperiodic in } x_l, \\ u^\alpha(x_l, x_0) \text{ satisfies } \alpha\text{-quasiperiodic radiation condition as } |x_0| \to \infty. \end{cases}$$

- Spectrum σ: parameterised by the spectra $\sigma(\alpha)$, $\alpha \in Y^*$, of the Helmholtz resonance problem, which in turn are known to consist of discrete values $\omega = \omega_i^\alpha$:

$$\sigma = \bigcup_{\alpha \in Y^*} \sigma(\alpha), \quad \sigma(\alpha) = \bigcup_{i=1}^{\infty} \omega_i^\alpha.$$

- Subwavelength part of the spectrum: resonant frequencies $\omega_i^\alpha \to 0$ as $\delta \to 0$.

Capacitance formulation for infinite periodic systems

- Generalised quasiperiodic capacitance matrix for $\alpha \neq 0$:

$$\mathcal{C}_{ij}^{\alpha} = \frac{\delta_i v_i^2}{|D_i|} C_{ij}^{\alpha}, \quad C_{ij}^{\alpha} := \int_{Y \setminus D} \overline{\nabla V_i^{\alpha}} \cdot \nabla V_j^{\alpha} \, \mathrm{d}x, \quad i, j = 1, \ldots, N;$$

- $V_i^{\alpha}, \; i = 1, \ldots, N$, solutions

$$\begin{cases} \Delta V_i^{\alpha} = 0 & \text{in } Y \setminus D, \\ V_i^{\alpha} = \delta_{ij} & \text{on } \partial D_j, \\ V_i^{\alpha}(x + l) = e^{\mathrm{i}\alpha \cdot l} V_i^{\alpha}(x) & \forall l \in \Lambda, \\ V_i^{\alpha}(x) \to 0 & \text{as } |x_0| \to \infty, \end{cases}$$

with $x = (x_l, x_0)$.

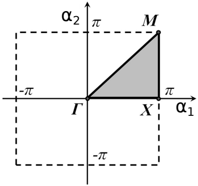
- $|\alpha| \neq 0$ fixed; $\delta \to 0$:

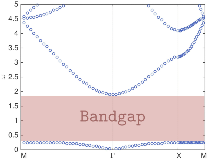
$$\omega_n^{\alpha} = \sqrt{\lambda_n^{\alpha}} + \mathcal{O}(\delta^{3/2}), \quad n = 1, \ldots, N;$$

- $\{\lambda_n^{\alpha} : n = 1, \ldots, N\}$: eigenvalues of $\mathcal{C}^{\alpha} \in \mathbb{C}^{N \times N}$, which satisfy $\lambda_n^{\alpha} = \mathcal{O}(\delta)$ as $\delta \to 0$.
- Resonant modes $u_n^{\alpha} \Leftarrow \mathbf{v}_n^{\alpha}$: eigenvector of $\mathcal{C}^{\alpha}$.
- Reciprocity: if $\forall \, \alpha \in Y^*$, the set of quasifrequencies at $\alpha \; = \;$ the one at $-\alpha$.

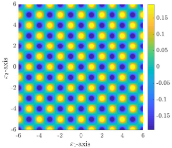
Subwavelength bandgap opening

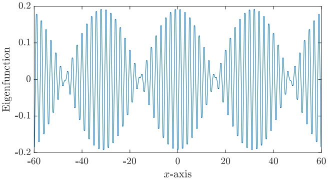
- Subwavelength bandgap opening in square crystals:





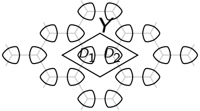
- Two-scale behaviour of the resonant mode for α close to (π, π): rapidly oscillating on the crystal scale, and a large-scale envelope which satisfies a homogenised equation.

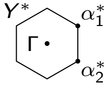




Honeycomb lattice of subwavelength resonators

- Honeycomb lattice:



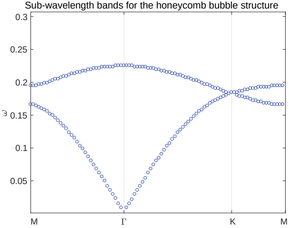


- At $\alpha = \alpha^*$, the first eigenfrequency $\omega^* := \omega(\alpha^*)$ of multiplicity 2.
- Conical behavior of subwavelength bands: The first band and the second band form a Dirac cone at α^*,

$$\omega_1(\alpha) = \omega(\alpha^*) - \lambda|\alpha - \alpha^*|[1 + \mathcal{O}(|\alpha - \alpha^*|)],$$
$$\omega_2(\alpha) = \omega(\alpha^*) + \lambda|\alpha - \alpha^*|[1 + \mathcal{O}(|\alpha - \alpha^*|)];$$

$\lambda = |c|\sqrt{\delta}\lambda_0 \neq 0$; $c = \partial C^{\alpha_{12}}/\partial \alpha_1\big|_{\alpha=\alpha^*}$;
$\lambda_0 = (1/2)\sqrt{v_1^2/(|D_1|C_{11}^{\alpha^*})}$.

- Dirac point at $\alpha = \alpha^*$.



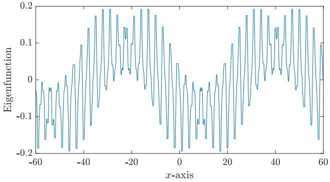
Honeycomb lattice of subwavelength resonators

- s : size of the unit cell; For α close to α^*, eigenmodes:

$$\tilde{u}_1(x)S_1(\tfrac{x}{s}) + \tilde{u}_2(x)S_2(\tfrac{x}{s}) + \mathcal{O}(\delta + s);$$

- Effective equation: $\tilde{u}_j$ satisfies

$$|c|^2\lambda_0^2\Delta\tilde{u}_j + \underbrace{\frac{(\omega-\omega^*)^2}{\delta}}_{\text{near zero}}\tilde{u}_j = 0.$$



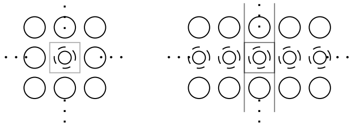
- Dirac equation:

$$\lambda_0\begin{bmatrix} 0 & (-ci)(\partial_1 - i\partial_2) \\ (-\bar{c}i)(\partial_1 + i\partial_2) & 0 \end{bmatrix}\begin{bmatrix}\tilde{u}_1 \\ \tilde{u}_2\end{bmatrix} = \frac{\omega-\omega^*}{\sqrt{\delta}}\begin{bmatrix}\tilde{u}_1 \\ \tilde{u}_2\end{bmatrix}.$$

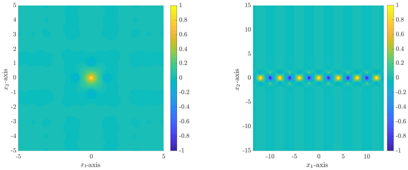
- Zero-phase shift propagation.
- Time-evolution of wave packets spectrally concentrated near conical points: time-dependent effective Dirac system.

Subwavelength trapping and guiding of waves

- Introduce a defect to a periodic arrangement of subwavelength resonators.

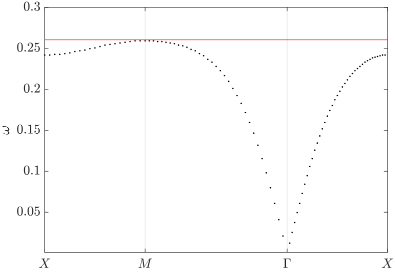


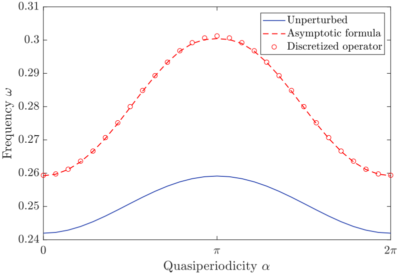
- Create a defect mode or a defect band inside the subwavelength band gap of the unperturbed structure.
- Defect band within the subwavelength band gap: large perturbation of the radius; localised to and guided along the line defect $\Leftarrow$ Absence of bound modes.



Topological interface defects

- Sensitivity to imperfections in the crystal's design:

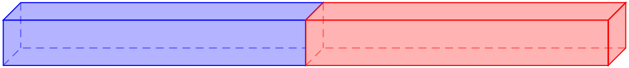




- Goal: design subwavelength wave guides whose properties are robust with respect to imperfections.
- Idea: Topological invariant which captures the crystal's wave propagation properties.

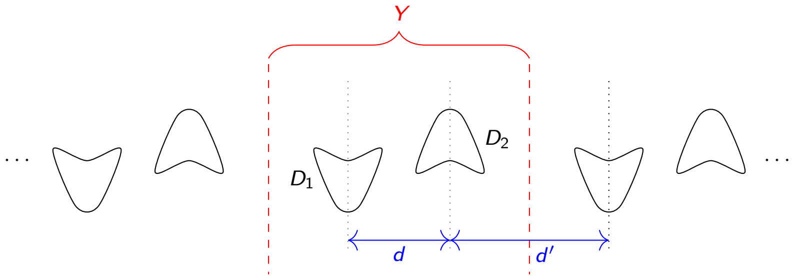
Topological interface defects

- Bulk-boundary correspondence:
    - Take two crystals with topologically different wave propagation properties (different values of the topological invariant);
    - Join half of crystal A to half of crystal B;
    - At the interface, a topologically protected interface mode will exist.



Topological interface defects

- An infinite chain of resonator dimers:[1]



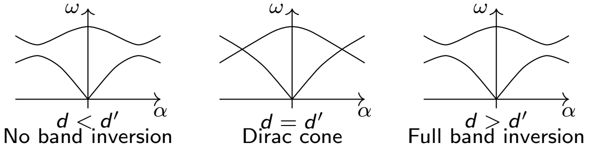
Two assumptions of geometric symmetry:

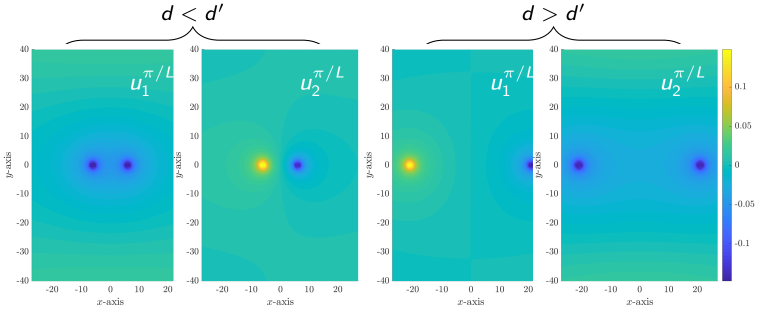
- Dimer is symmetric, in the sense that $D(:= D_1 \cup D_2) = -D$;
- Each resonator has reflective symmetry.

[1] Analogue of the Su-Schrieffer-Heeger model in topological insulator theory in quantum mechanics.

Topological interface defects

- Band inversion occurs between $d < d'$ and $d > d'$; Monopole/dipole natures of the 1st and 2nd eigenmodes have swapped between $d < d'$ and $d > d'$ regimes; Dirac degeneracy precisely when $d = d'$.





Topological interface defects

- Change in the argument of C_{12}^{α} as α varies over Y^* quantised:

$$\varphi_n^z = -\frac{1}{2}\left[\arg(C_{12}^{\alpha})\right]_{Y^*}.$$

- The Zak phase:

$$\varphi_n^z := \int_{Y^*} A_n(\alpha)\, d\alpha; \quad Y^* = \mathbb{R}/2\pi\mathbb{Z} \simeq (-\pi, \pi] \quad \text{(first Brillouin zone)};$$

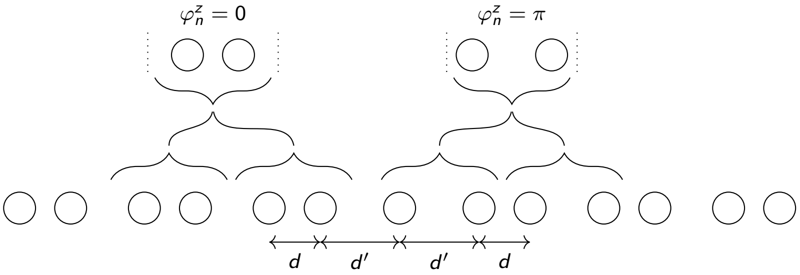
- Berry-Simon connection:

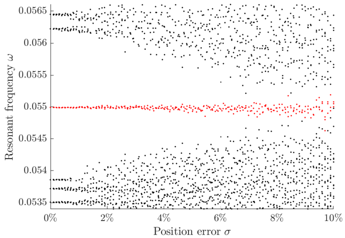
$$A_n(\alpha) := i \int_D u_n^{\alpha} \frac{\partial}{\partial \alpha} \overline{u}_n^{\alpha}\, dx; \quad n = 1, 2.$$

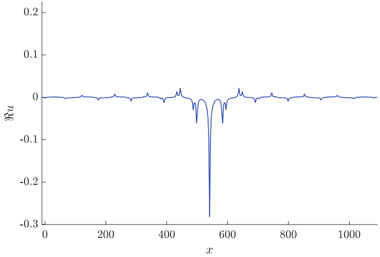
- The cases $d > d'$ and $d < d'$ have different Zak phases.

Topological defects

- Finite chain of resonators: robust localised eigenmode

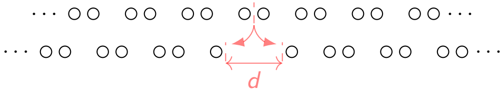




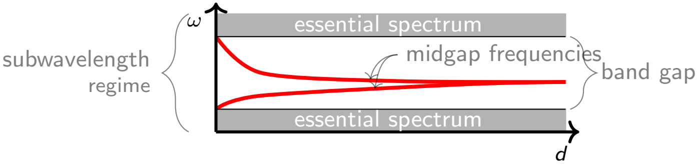


Edge modes in a dislocated chain

- Introduce a dislocation (with size $d > 0$) in an array of pairs of subwavelength resonators having subwavelength band gap to create midgap frequencies.

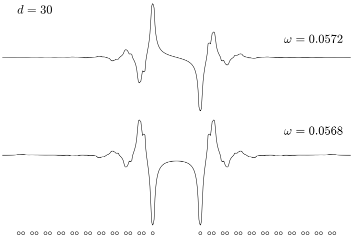


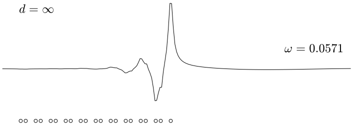
- As d increases, a midgap frequency appears from each edge of the subwavelength band gap. These two frequencies converge to a single value within the subwavelength band gap as $d \to \infty$.



Topological properties of Hermitian systems

- Two edge modes for an array of 42 spherical resonators of radius 1; edge mode of the corresponding 'half system':





Anderson localisation

- Strong localisation in randomly perturbed systems with long-range interactions.
- Λ: lattice of dimension $1 \leq d_l \leq d$;

$$\mathcal{D} = \bigcup_{m\in\Lambda} \bigcup_{i\in\{1,\ldots,N\}} D_i^m, \qquad D_i^m = D_i + m, \qquad D = \bigcup_{i\in\{1,\ldots,N\}} D_i;$$

- Real-space capacitance matrix:

$$\widehat{\mathcal{C}}^m = \mathcal{U}^{-1}[\mathcal{C}^\alpha](m), \quad m \in \Lambda.$$

- Discrete Floquet transform:

$$\mathcal{U}[\phi](\alpha) := \sum_{m\in\Lambda} \phi(m) e^{\mathrm{i}\alpha\cdot m}, \qquad \mathcal{U}^{-1}[\psi](m) := \frac{1}{|Y^*|} \int_{Y^*} \psi(\alpha) e^{-\mathrm{i}\alpha\cdot m}\, \mathrm{d}\alpha.$$

- Characterisation of localisation:

$$\mathcal{B}_m \sum_{n\in\Lambda} \mathcal{C}^{m-n} \mathbf{u}^n = \omega^2 \mathbf{u}^m,$$

for every $m \in \Lambda$ (real-space variable); $\mathbf{u}^m \in \mathbb{R}^N$; $\mathcal{B}_m$: $N \times N$ diagonal matrix whose i^{th} entry is given by $b_i^m = 1 + x_i^m$; x_i^m: random perturbation of the material parameter of the resonator i in the cell m; uniform distribution $U[x - \sqrt{3}\sigma, x + \sqrt{3}\sigma]$.

Laurent-operator formulation

- If $\Lambda = \mathbb{Z}$,

$$\mathfrak{B}\mathfrak{C}\mathfrak{u} = \omega^2 \mathfrak{u}.$$

- Doubly infinite matrices and vectors:

$$\mathfrak{C} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & \mathcal{C}^0 & \mathcal{C}^1 & \mathcal{C}^2 & \mathcal{C}^3 & \cdots \\ \cdots & \mathcal{C}^{-1} & \mathcal{C}^0 & \mathcal{C}^1 & \mathcal{C}^2 & \cdots \\ \cdots & \mathcal{C}^{-2} & \mathcal{C}^{-1} & \mathcal{C}^0 & \mathcal{C}^1 & \cdots \\ \cdots & \mathcal{C}^{-3} & \mathcal{C}^{-2} & \mathcal{C}^{-1} & \mathcal{C}^0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathfrak{u} = \begin{pmatrix} \vdots \\ \mathbf{u}^{-1} \\ \mathbf{u}^0 \\ \mathbf{u}^1 \\ \mathbf{u}^2 \\ \vdots \end{pmatrix}, \quad \mathfrak{B} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & \mathcal{B}_{-1} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & \mathcal{B}_0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \mathcal{B}_1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \mathcal{B}_2 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

- $\mathfrak{C}$: (block) Laurent operator corresponding to the symbol $\mathcal{C}^\alpha$.
- A localised mode corresponds to an eigenvalue of the operator $\mathfrak{B}\mathfrak{C}$.
- In the periodic case (when $\mathfrak{B} = I$), the spectrum of the Laurent operator $\mathfrak{C}$ is continuous and does not contain eigenvalues, so there are no localised modes.
- The operator $\mathfrak{B}\mathfrak{C}$ might have a pure-point spectrum in the non-periodic case.

Toeplitz matrix formulation for compact defects

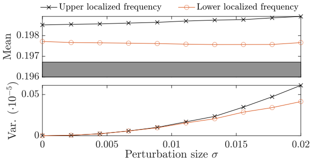
- Compact defects: $\mathcal{B}_m$ are identity for all but finitely many m; $0 \leq m \leq M$.
- X_m: diagonal matrix with entries x_i^m.
- (Block) Toeplitz matrix formulation: ω corresponds to a localised mode iff

$$\det\big(I - \mathcal{X}\mathcal{T}(\omega)\big) = 0.$$

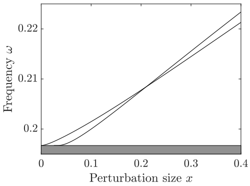
- $\mathcal{X}$: block-diagonal matrix with entries X_m;
- $$\mathcal{T}(\omega) = \begin{pmatrix} T^0 & T^1 & T^2 & \cdots & T^M \\ T^{-1} & T^0 & T^1 & \cdots & T^{M-1} \\ T^{-2} & T^{-1} & T^0 & \cdots & T^{M-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T^{-M} & T^{-(M-1)} & T^{-(M-2)} & \cdots & T^0 \end{pmatrix};$$
- $$T^m = -\frac{1}{|Y^*|}\int_{Y^*} e^{i\alpha m}\mathcal{C}^\alpha \left(\mathcal{C}^\alpha - \omega^2 I\right)^{-1} d\alpha.$$

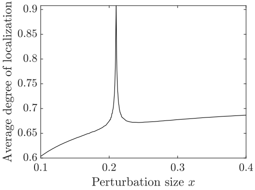
Level repulsion and hybridisation

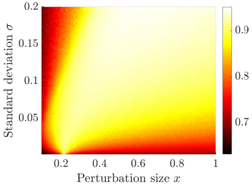
- Level repulsion: random perturbations ⇒ average value of each midgap frequency to move further apart and further apart the edge of the band gap



- Hybridisation ⇒ sharp peak at the transition point in the degree of localisation



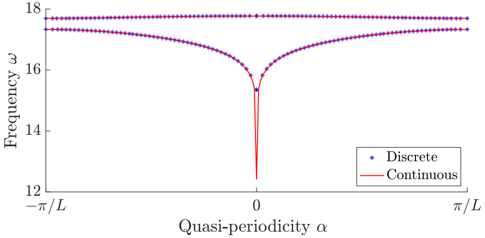


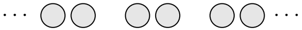


Spectral convergence in large finite resonator arrays

- Pointwise convergence to the essential spectrum: Any eigenvalue/eigenvector of $\mathcal{C}^\alpha$ can be approximated by eigenvalues/eigenvectors of $\mathcal{C}_\mathrm{f}$; Converse not true: edge effect $\Rightarrow$ greatest effect on eigenmodes within the first radiation continuum.
- Convergence in distribution of the discrete density of states for the finite M-system of N periodically repeated resonators to the (continuous) density of states of the infinite system:

$$D_\mathrm{f}(\omega) := \frac{1}{MN}\sum_{j=1}^{MN} \delta\left(\omega - \omega_j^{(M)}\right) \to D(\omega) := \frac{1}{N}\sum_{k=1}^{N}\int_{Y^*} \delta\left(\omega - \hat{\omega}_k(\alpha)\right) d\alpha.$$





Spectral convergence in large finite resonator arrays

- Weak convergence of $\mathcal{C}_{\mathrm{f}}$ ($M \times M$-block matrix with blocks of size N) to corresponding (translationally invariant) Toeplitz matrix $\mathcal{C}_{\mathrm{t}}$ of the infinite structure with symbol $\mathcal{C}^\alpha$:

$$\mathcal{C}_{\mathrm{t}} = \begin{pmatrix} \mathcal{C}^0 & \mathcal{C}^1 & \cdots & \mathcal{C}^M \\ \mathcal{C}^{-1} & \mathcal{C}^0 & \cdots & \mathcal{C}^{M-1} \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{C}^{-M} & \mathcal{C}^{1-M} & \cdots & \mathcal{C}^0 \end{pmatrix}.$$

- $\mathcal{C}_{\mathrm{f}}$, $\mathcal{C}_{\mathrm{t}}$ asymptotically equivalent: $\frac{1}{\sqrt{M}}\|\mathcal{C}_{\mathrm{f}} - \mathcal{C}_{\mathrm{t}}\|_F \to 0$; $\|\mathcal{C}_{\mathrm{f}}\|_2, \|\mathcal{C}_{\mathrm{t}}\|_2$ uniformly bounded.
- $\mathcal{C}_{\mathrm{f}}, \mathcal{C}_{\mathrm{t}}$: identical eigenvalue distributions as their sizes $\to \infty$.

Spectral convergence in large finite resonator arrays

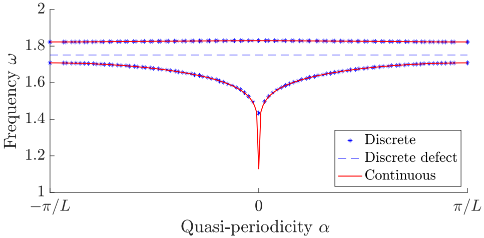
- Truncated Floquet transform: $(\omega_j, u_j), (u_j)_m$: vector of length N associated to cell $m \in \Lambda$;

$$(\widehat{u}_j)_\alpha = \sum_{m \in \text{finite lattice}} (u_j)_m e^{i\alpha \cdot m}; \quad \alpha_j = \arg\max_{\alpha \in Y^*} \|(\widehat{u}_j)_\alpha\|_2.$$

- Defect modes in infinite systems of resonators have corresponding modes in finite systems which converge as the size of the system increases; Rate of convergence in terms of the length of the truncated structure:

$$d_l = d \Rightarrow \text{exponential}; \quad d_l < d \Rightarrow \text{algebraic}.$$

- Principle applicable to structures that are not translationally invariant:



Disordered systems

- Broken translation invariance: globally or locally;
- Particular classes: quasiperiodic systems; hyperuniform systems; random block systems.
- Random block systems: Sample S (single resonator block) and D (dimer resonator block) with resp. probability p_S and p_D: $SSSSDSSSSSDSSS$.
- Obtain Bloch band functions for a disordered block structure by imposing quasiperiodic boundary conditions on the respective finite systems;
- Compute the eigenvalues of the quasiperiodic capacitance matrix $C^\alpha : (C^\alpha_{ij}) := -\int_{\partial D_i} \frac{\partial V_j^\alpha}{\partial \nu} \, d\sigma;$
- N band functions of the system: given by the Bloch band functions $\alpha \in Y^* \mapsto \omega_p(\alpha)$ mapping a quasiperiodicity α in the Brillouin zone to the p^{th} eigenvalue of C^α.
- Convergence of empirical cumulative density functions under increasing system size (Jacobi operator; metric transitivity $\Rightarrow$ ergodicity)

Disordered systems

- Average shift in the band function frequency:

$$\delta\lambda_i := \frac{1}{2\pi L} \int_{\alpha \in Y^*} |\lambda_i(\alpha) - \lambda_i(0)| d\alpha.$$

- Thouless ratio $g(\lambda_i)$ for a given frequency λ_i of $\mathcal{C}$:

$$g(\lambda_i) := \frac{\delta\lambda_i}{\Delta(\lambda_i)} \quad \text{with} \quad \Delta(\lambda_i) = \frac{1}{D(\lambda_i)L};$$

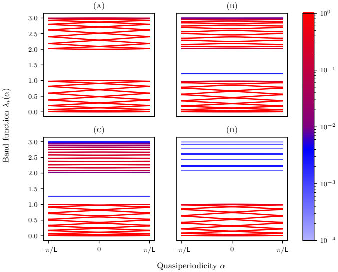
  $D(\lambda)$: density of states (computed using a Gaussian kernel density estimate on the eigenvalues $\lambda_1, \ldots, \lambda_N$ of $\mathcal{C}$); L: physical length of the system.

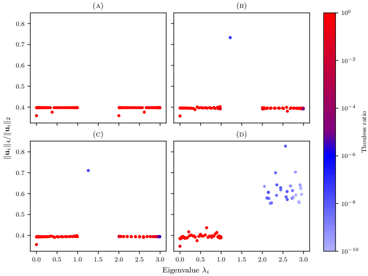
- Thouless criterion of localisation:

$$\boldsymbol{u}_i \text{ is } \begin{cases} \text{delocalised} & \text{if } g(\lambda_i) \approx 1, \\ \text{localised} & \text{if } g(\lambda_i) \ll 1. \end{cases}$$

Disordered systems

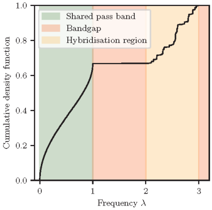
- (A) periodic, (B) SSH, (C) dislocated, and (D) random block system.

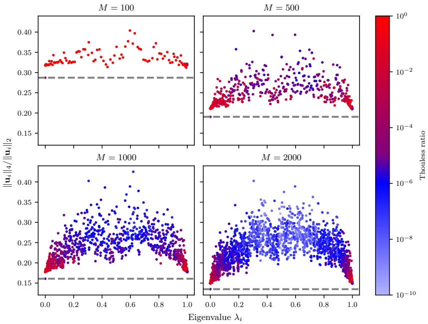




Block disordered systems

- Band iff it is in the band of all constituent blocks;
- Bandgap *iff* it is in the bandgap of all constituent blocks;
- Hybridisation region otherwise.





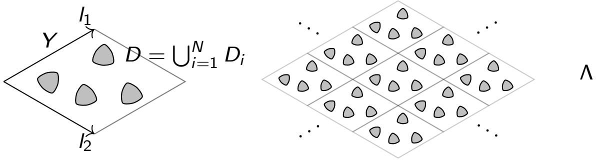
Capacitance formulation for space-time modulated systems

- Wave equation in a space-time modulated systems:

$$\left(\frac{\partial}{\partial t}\frac{1}{\kappa(x,t)}\frac{\partial}{\partial t} - \nabla\cdot\frac{1}{\rho(x,t)}\nabla\right)u(x,t) = 0, \quad x\in\mathbb{R}^d, t\in\mathbb{R}.$$

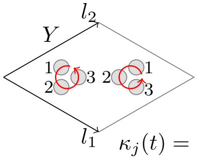
- Y: unit cell; $\mathcal{D} = \bigcup_{m\in\Lambda} D + m$; $\mathcal{D}_i = \bigcup_{m\in\Lambda} D_i + m$; $D_i, i = 1,\ldots,N$.
- Time-modulation of the resonators:

$$\kappa(x,t) = \begin{cases}\kappa, & x\in\mathbb{R}^d\setminus\overline{\mathcal{D}},\\ \kappa_r\kappa_i(t), & x\in\mathcal{D}_i,\end{cases}, \qquad \rho(x,t) = \begin{cases}\rho, & x\in\mathbb{R}^d\setminus\overline{\mathcal{D}},\\ \rho_r\rho_i(t), & x\in\mathcal{D}_i.\end{cases}$$

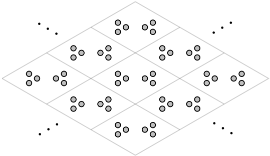


Capacitance formulation for space-time modulated systems

- Reciprocity: if $\forall\, \alpha \in Y^*$, the set of quasifrequencies at $\alpha \;=\;$ the one at $-\alpha$.
- Folding of the static band structure might create degenerate points;
- Time modulations break the time-reversal symmetry and open degenerate points of the folded band structure into non-symmetric bandgaps; opposite propagation directions: distinct bandgaps.
- Phase-shifted ("rotation like") time modulations of subwavelength resonators can provide a kind of "artificial spin".
- Trimer honeycomb lattice with phase-shifted time-modulations inside the trimers:

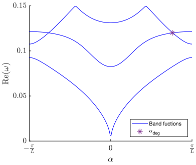


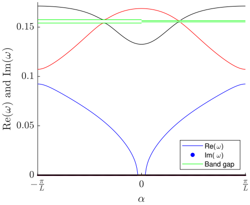
$$\kappa_j(t) = 1 + \epsilon \sin\left(\Omega t + \frac{2\pi j}{3}\right)$$

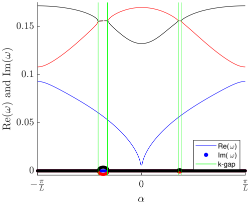


Capacitance formulation for space-time modulated systems

- Non-reciprocal wave propagation and k-gaps







- Non-symmetric bandgaps $\Rightarrow$ unidirectional excitation of the operating waves;
- Existence of k-gaps $\Rightarrow$ exponentially growing wave propagation.

Non-Hermitian skin effect

- PDE model: $D = \cup_{i=1}^N$ chain of finitely many periodic resonators (in x_1-direction) with a non-Hermitian imaginary gauge potential

$$\begin{cases} \Delta u + \omega^2 \dfrac{\rho}{\kappa} u = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D}, \\ \Delta u + \omega^2 \dfrac{\rho_i}{\kappa_i} u + \gamma \partial_{x_1} u = 0 \quad \text{in} \quad D_i, i = 1, \ldots, N, \\ u|_+ = u|_- \quad \text{on} \quad \partial D_i, \\ \dfrac{\rho_i}{\rho} \dfrac{\partial u}{\partial \nu}\Big|_+ = \dfrac{\partial u}{\partial \nu}\Big|_- \quad \text{on} \quad \partial D_i, \\ u \text{ satisfies the radiation condition.} \end{cases}$$

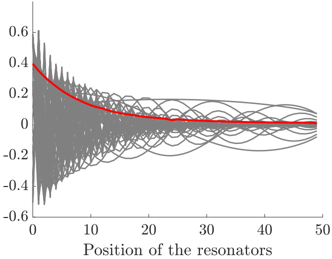
- Eigenmodes and eigenfrequencies approximated by the eigenvectors and square roots of the eigenvalues of the gauge capacitance matrix:

$$(\mathcal{C}_N^\gamma)_{i,j} = - \frac{\delta_i v_i^2}{\int_{D_i} e^{\gamma x_1} \, \mathrm{d}x} \int_{\partial D_i} e^{\gamma x_1} \frac{\partial V_j}{\partial \nu} \mathrm{d}\sigma(x).$$

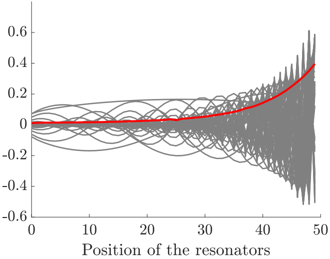
- Condensation of bulk eigenmodes at one of the edges of the system (depending on $\mathrm{sign}(\gamma)$) as its size increases.
- “Infinite” order exceptional point.

Non-Hermitian skin effect

- Eigenvectors of the gauge capacitance matrix are exponentially decaying or growing, depending on the sign of γ:



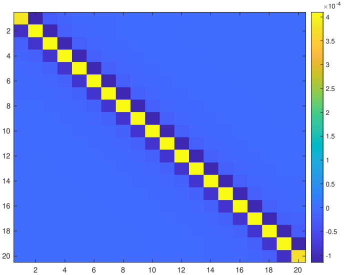
$\gamma = 1.$



$\gamma = -1.$

Non-Hermitian skin effect

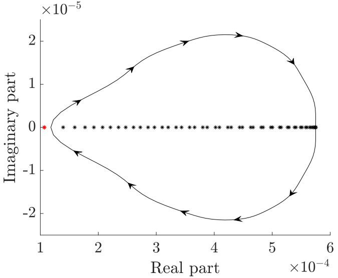
- Gauge capacitance matrix $\mathcal{C}^\gamma$: perturbed Toeplitz structure $\Leftarrow$ system: almost translational invariant; dense $\Leftarrow$ long-range coupling;
- $\mathcal{C}^\gamma$: approximated by a banded Toeplitz matrix with a perturbation on the edge.
- Symbol function: $a(z) = \sum_{j=-(k-1)}^{k-1} a_j z^j$.



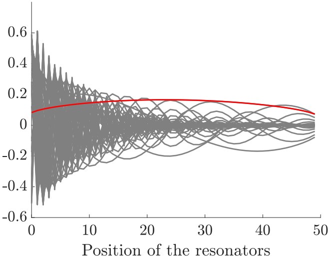
$$\approx \begin{bmatrix} a_0 & \cdots & a_{-k-1} & 0 & \cdots & 0 \\ \vdots & a_0 & \ddots & \ddots & & \vdots \\ a_{k+1} & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & a_{-k-1} \\ \vdots & & \ddots & \ddots & a_0 & \vdots \\ 0 & \cdots & 0 & a_{k+1} & \cdots & a_0 \end{bmatrix}$$

Non-Hermitian skin effect

- Define $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and $I(a(\mathbb{T}), \lambda)$ the winding number of $a(\mathbb{T})$ at λ in the positive direction.
- Exponential decay of the pseudo-eigenvectors: predicted by the winding number. Topological protection of the associated (real) eigenfrequencies.



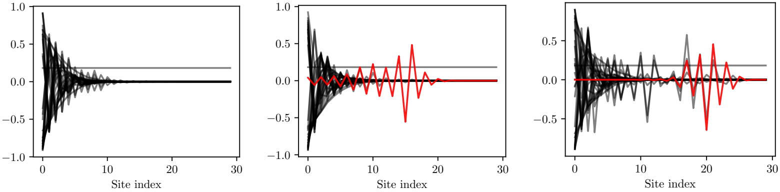
$a(\mathbb{T})$ and the eigenvalues.



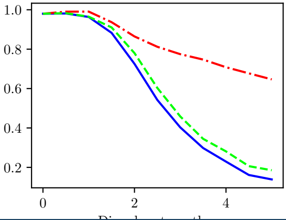
Eigenmodes.

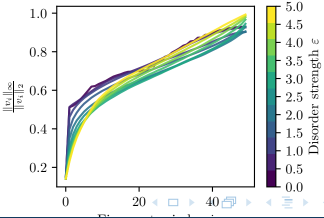
Stability of the non-Hermitian skin effect

- Single realisations with increasing disorder strengths:



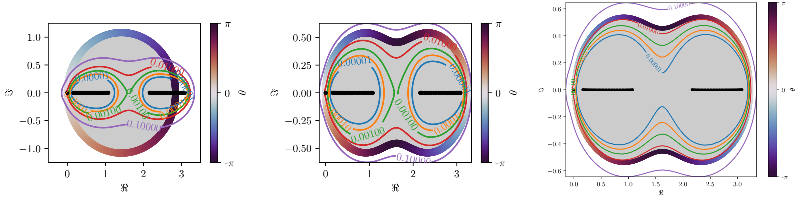
- Competition between the non-Hermitian skin effect and the disorder-induced Anderson localisation;
- As the strength of the disorder increases, more and more eigenmodes become localised in the bulk.





Dimer systems

- Dimer systems $\Rightarrow$ Perturbed Block Toeplitz matrices.
- Fredholm index of the associated operator ($=$ winding of the determinant of its symbol) takes value zero at some point on the unit circle.
- Winding of the two eigenvalues of the symbol: predicts accurately the exponential decay of the eigenmodes and is the limit of the pseudospectrum as $N \to \infty$.

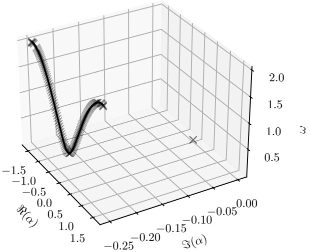


Convergence results for non-Hermitian large systems

- Spectrum of the limiting operator: Non-Bloch eigenmodes $\Rightarrow$ generalised (complex) Brillouin zone

  $\mathcal{Y}^* := \{(\alpha, \beta(\alpha)) \in Y^* \times \mathbb{R} : \lambda^{\alpha+i\beta(\alpha)} \in \mathbb{R}^+\}$; $\lambda^{\alpha+i\beta(\alpha)}$ eigenvalue of $\mathcal{C}^{\gamma,\alpha+i\beta(\alpha)}$.

- Convergence to the complex band structure:



- Systems with complex material parameters can be reduced to Hermitian systems away from their exceptional points.
- Non-Hermitian systems with imaginary gauge potentials / Non-Hermitian systems with complex material parameters: fundamentally distinct.

Generalised Brillouin zone

- $T(a)$: tridiagonal k-Toeplitz operator with symbol $a(z)$ and non-zero off-diagonal entries

$$a : z \mapsto \begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 & c_k z \\ c_1 & a_2 & b_2 & & & 0 \\ 0 & c_2 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & b_{k-2} & 0 \\ 0 & & & c_{k-2} & a_{k-1} & b_{k-1} \\ b_k z^{-1} & 0 & \cdots & 0 & c_{k-1} & a_k \end{pmatrix}.$$

- Non-reciprocity rate:

$$\Delta = \ln \prod_{j=1}^{k} \left|\frac{b_j}{c_j}\right|.$$

- Generalised Brillouin zone:

$$\mathcal{B} = \left\{ \alpha + \mathrm{i}\beta \mid \alpha \in [-\pi, \pi), \beta \in [0, \Delta] \right\};$$

- $\sigma(T(a)) = \bigcup_{\alpha+\mathrm{i}\beta\in\mathcal{B}} \sigma(a(e^{-\mathrm{i}(\alpha+\mathrm{i}\beta)}))$, up to at most $(k-1)$ points which may be in $\sigma(T(a))$ but not in $\bigcup_{\alpha+\mathrm{i}\beta\in\mathcal{B}} \sigma(a(e^{-\mathrm{i}(\alpha+\mathrm{i}\beta)}))$.

Generalised Brillouin zone

- **Open boundary conditions** $\Rightarrow$ k-Toeplitz matrix T_{mk} of order mk associated with the symbol a:

$$\lim_{m\to\infty} \sigma(T_{mk}(a)) = \bigcup_{\alpha\in Y^*} \sigma(a(e^{-\mathrm{i}(\alpha+\mathrm{i}\Delta/2)})).$$

- T_{mk} symmetrised $\Rightarrow$ $T(\tilde{a})_{mk}$; $\tilde{a}$: collapsed symbol (no winding).
- **Periodic boundary conditions** $\Rightarrow$ tridiagonal k-circulant matrix:

$$(C_{mk}(a))_{ij} := \begin{cases} c_k & i=0, j=mk, \\ b_k & i=mk, j=0, \\ (T_{mk}(a))_{ij} & \text{otherwise.} \end{cases}$$

- Laurent operator L associated with the symbol a:

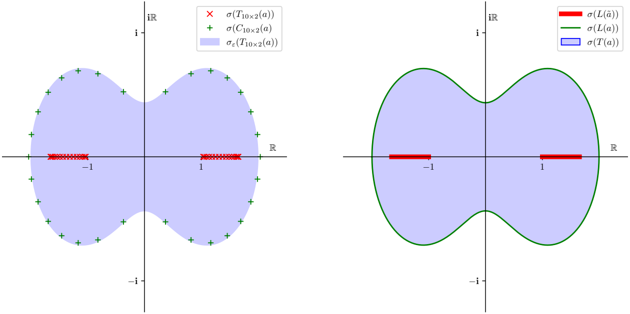
$$\sigma(C_{mk}(a)) = \bigcup_{j=0}^{m-1} a(e^{2\pi \mathrm{i} j/m}) \to \bigcup_{\alpha\in Y^*} a(e^{-\mathrm{i}\alpha}) = \sigma(L(a)).$$

- Convergence of the pseudo-spectrum:

$$\lim_{m\to\infty} \sigma_\epsilon(T_{mk}(a)) = \sigma_\epsilon(T(a)); \quad \lim_{\epsilon\to 0}\lim_{m\to\infty} \sigma_\epsilon(T_{mk}(a)) = \lim_{\epsilon\to 0} \sigma_\epsilon(T(a)) = \sigma(T(a)).$$

Generalised Brillouin zone

- **Three spectral limits:**
  (i) Eigenvalues of the circulant matrix $C_{mk}(a)$ arrange around the symbol curve $\Rightarrow$ converge to the Laurent operator limit $L(a)$;
  (ii) Eigenvalues of the Toeplitz matrix $T_{mk}(a)$ arrange around the collapsed symbol $\tilde{a}$ $\Rightarrow$ converge to the Laurent operator $T(\tilde{a})$;
  (iii) ε-pseudospectrum of $T_{mk}(a)$ corresponds exactly to the interior of the symbol curve $\Rightarrow$ converges to the actual Toeplitz limit $T(a)$.



Concluding remarks

- **Mathematical foundations** of **subwavelength physics**:
  - Localisation and topological properties of reciprocal and non-reciprocal systems of subwavelength resonators;
  - Non-reciprocity can be achieved by imaginary gauge potentials but also by time-modulations.
  - Dirac, exceptional point, and folding degeneracies.
  - Unified capacitance matrix framework for studying linear and nonlinear systems with long range interactions in three dimensions.
  - Interplay between nonlinearities and Dirac and exceptional point degeneracies.
  - In one dimension: short range interactions $\Rightarrow$ Haldane: "Quantum phenomena are not particular to quantum systems".