Speckle formation of laser light in random media
The Gaussian conjecture

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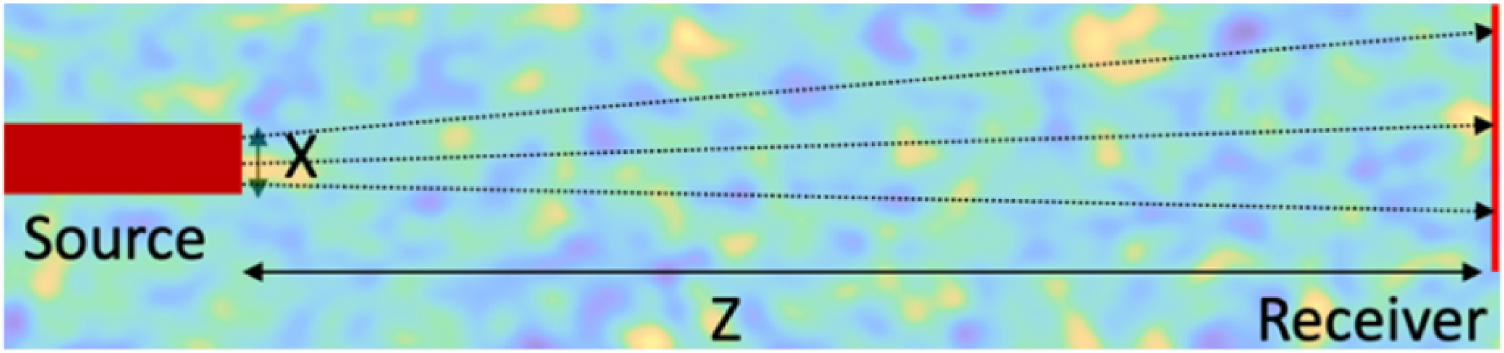
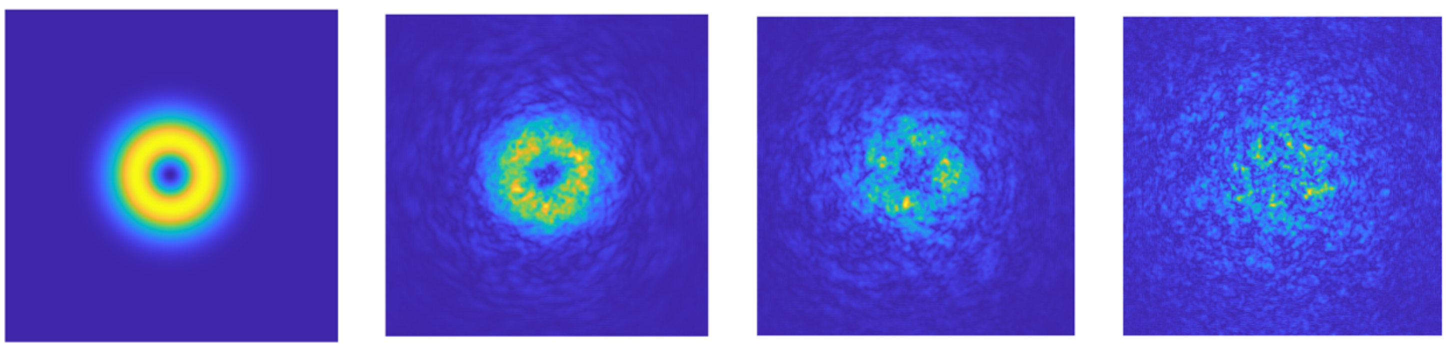
Joint work with Anjali Nair.

Wave beam propagation in random media

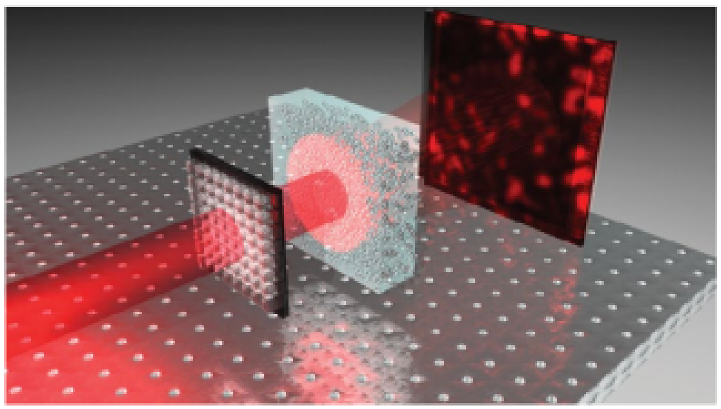


Wave-beam propagating in turbulent atmosphere

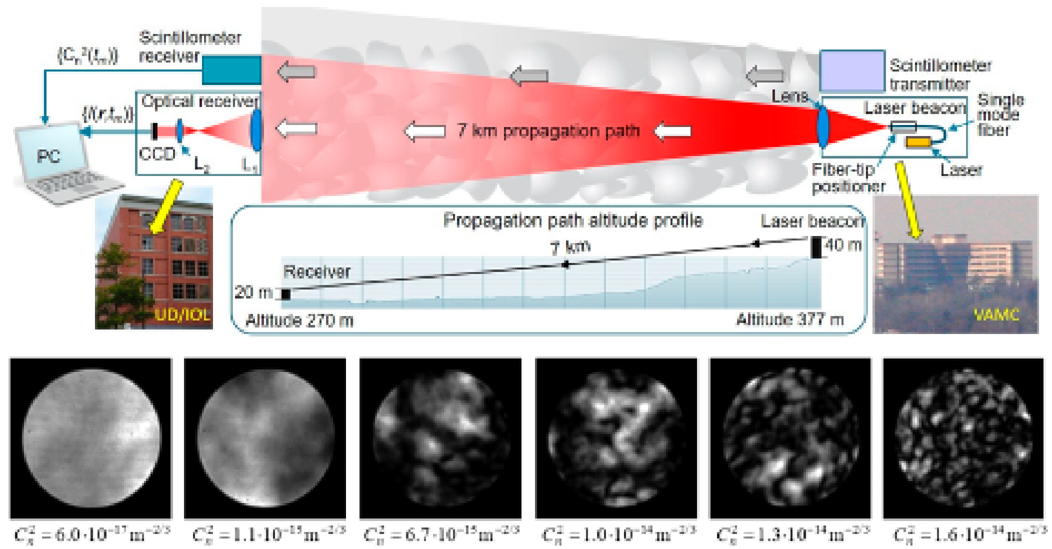


Receiver reading as turbulence strength increases.

Speckle formation



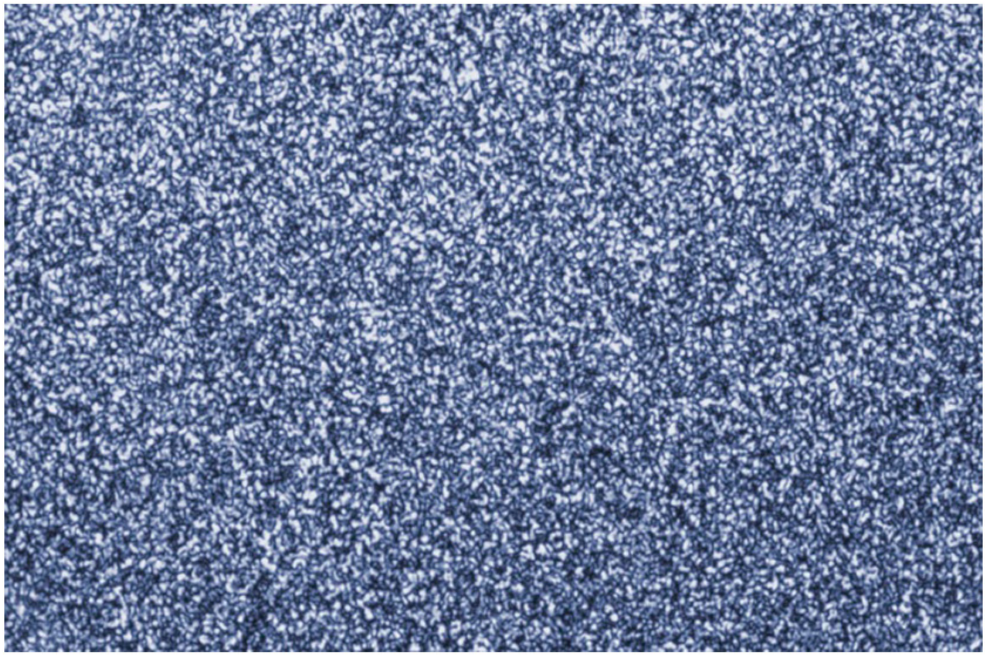
Ref: Cao, H., Mosk, A.P. and Rotter, S., *Nature Physics* (2022)



Ref: Vorontsov, A. M., Vorontsov, M. A., Filimonov, G. A., and Polnau, E. *Applied Sciences* (2020)

Speckle is manifestation of constructive/destructive interference.
Different applications for (narrow frequency band) laser light.

Speckle patterns



Heuristics of fully formed Speckle

Fully formed speckle with superposition of random plane waves.
Location $x \in \mathbb{R}^d$ fixed and:

$$u(x) = u_r + iu_i = \frac{1}{\sqrt{M}} \sum_{k=1}^{M} a_k e^{i\phi_k}, \qquad I(x) = |u(x)|^2$$

with ϕ_k iid uniform on $(0, 2\pi)$ and a_k iid mean zero with $\mathbb{E}a_k^2 = a^2$. Then

$$\mathbb{E}u_r = \mathbb{E}u_i = \mathbb{E}u_r u_i = 0, \qquad \mathbb{E}u_r^2 = \mathbb{E}u_i^2 = \frac{1}{2}a^2$$

and in limit $M \to \infty$,

$$\rho(u_r, u_i) = \frac{1}{\pi a^2} e^{-\frac{1}{a^2}(u_r^2 + u_i^2)}, \qquad \rho(I) = \frac{1}{a^2} e^{-\frac{I}{a^2}}, \quad \mathbb{E}I = \mathbb{E}I^2 = a^2.$$

- Exponential distribution of intensity corroborates observed speckle.
- (u_r, u_i) asymptotically complex circular Gaussian.
- a_k? ϕ_k? M? Correlations at different x?

[Goodman 19; Carminati-Schotland 21]

Heuristic Speckle Formation

Gaussian Conjecture: speckle patterns form after long-distance propagation as wavefield becomes complex circular Gaussian distributed:
Real and imaginary parts of field are mean-zero iid Gaussian *fields*.

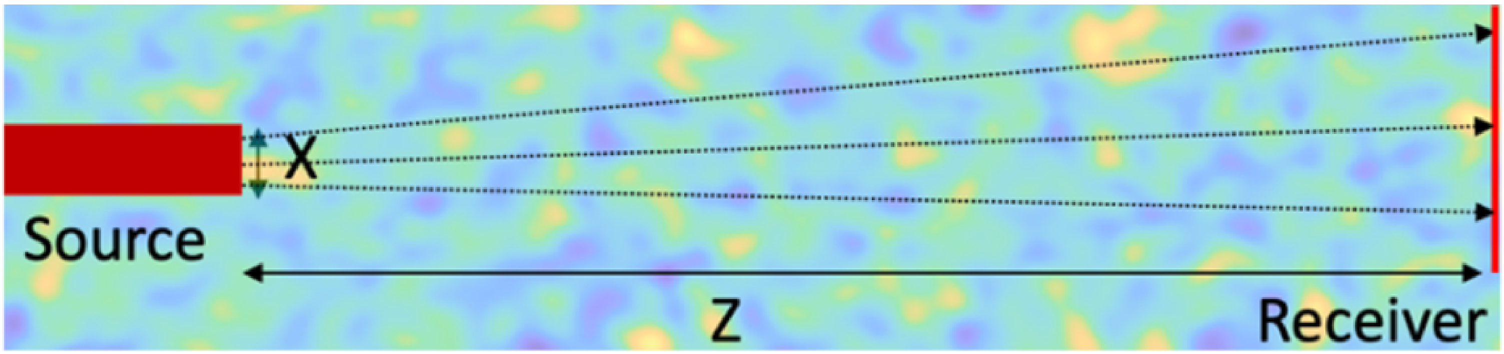
Conjecture settled in Itô-Schrödinger / paraxial regimes of wave propagation. Joint work with Anjali Nair.

B. Nair 2024. Complex Gaussianity of long-distance random wave processes. Arxiv arXiv:2402.17107

B. Nair 2024. Long distance propagation of light in random media with partially coherent sources. Arxiv arXiv:2406.05252.

B. Nair 2024. Long distance propagation of wave beams in paraxial regime. Arxiv arXiv:2409.09514

Wave beam propagation



Wave propagation in $z > 0$ with Helmholtz model ($-i\partial_t \to \omega = ck_0$):

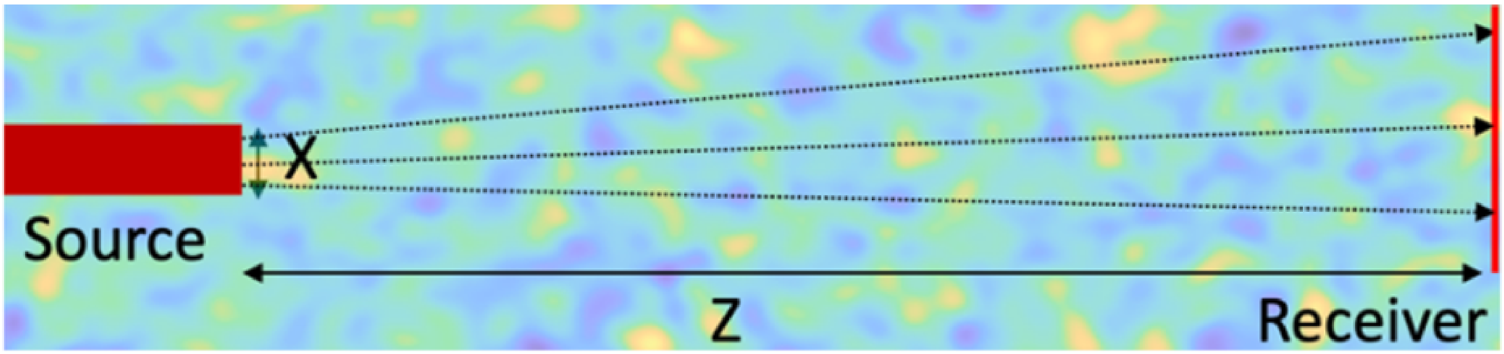
$$\left(\partial_z^2 + \Delta_x + k_0^2[1 + \nu(z,x)]\right) p(z,x) = \delta'(z) p_0(x).$$

$\nu(z,x)$ random perturbation of index of refraction.

$p_0(x)$ (deterministic) incident source profile.

- $x \mapsto p(z,x)$ for z large? Speckle? Scaling-dependent.

Scales and scalings



$$\left(\partial_z^2 + \Delta_x + k_0^2[1 + \nu(z,x)]\right) p(x,z) = \delta'(z) p_0(x).$$

Parameters:

l_0 correlation length $\nu = \nu(\frac{z}{l_0}, \frac{x}{l_0})$. $l_0 \approx 2\ 10^{-3} m$.

$k_0 = \lambda^{-1}$ with wavelength $\lambda \approx 10^{-6} m$. $\lambda \ll l_0$.

ω_0 width of incident source $p_0 = p_0(\frac{x}{w_0})$. $w_0 \approx 0.05 - 1m$.

$l_0 Z$ typical distance of interest, of order $10^3 m$.

Small parameters

l_0 correlation length $\nu = \nu(\frac{z}{l_0}, \frac{x}{l_0})$. $l_0 \approx 10^{-3}m$.
$k_0 = \lambda^{-1}$ with wavelength $\lambda \approx 10^{-6}m$.
ω_0 width of incident source $p_0 = p_0(\frac{x}{w_0})$. $w_0 \approx 0.05 - 1m$.
$l_0 Z$ typical distance of interest, of order $10^3 m$.

Define:

$$\theta = \frac{1}{k_0 l_0}, \quad \varepsilon = \frac{l_0}{w_0}, \quad \eta = \frac{Z}{k_0 w_0}, \quad \sigma^2 = \frac{w_0^2}{l_0^2 Z^3}.$$

Here $\sigma \approx |\nu| \approx 10^{-7} - 10^{-6}$ models fluctuation strength. We find

$$\theta \approx 10^{-3}, \quad \varepsilon \approx 10^{-3} - 10^{-1}, \quad \eta \approx 10^{-2} - 1.$$

- $(\theta, \varepsilon) \to 0$ model high frequency, weak-coupling, and *beam structure*.
- $\varepsilon, \theta \ll \eta = 1$ in kinetic and $\varepsilon, \theta \ll \eta \ll 1$ in diffusive regimes.
- Optical thickness of medium at z is $\frac{z}{\eta^2}$.

Wave beam propagation

$$\Big(\partial_z^2 + \Delta_x + k_0^2[1+\nu(z,x)]\Big)\; p(z,x) = \;\delta'(z)p_0(x).$$

With above parameter choices, wave equation along with

$$z \to \frac{\eta z}{\varepsilon\theta}, \quad x \to x, \quad k_0 \to \frac{k_0}{\theta}, \quad \nu \to \frac{\varepsilon^{\frac{1}{2}}\theta^{\frac{3}{2}}}{\eta^{\frac{3}{2}}}\nu$$

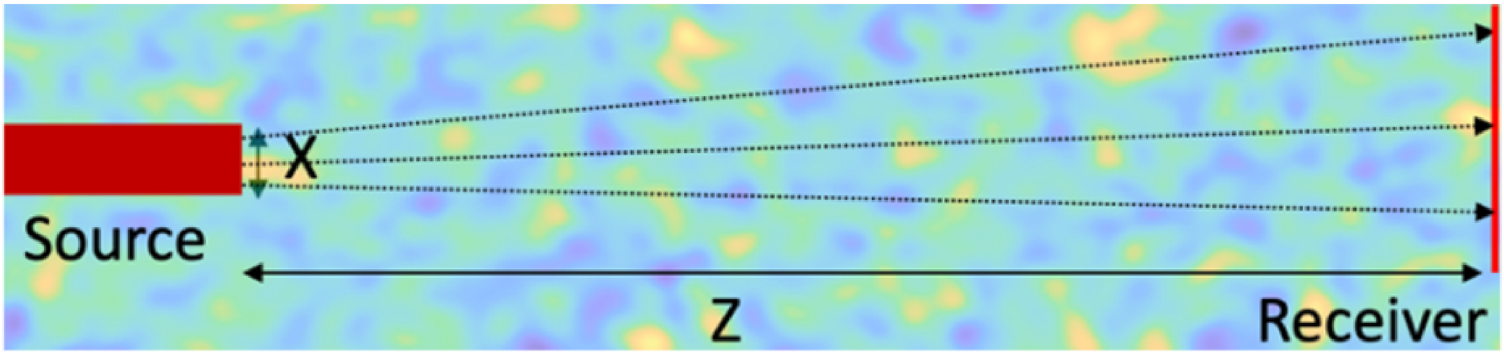
then *paraxial envelope* u given by

$$u(z,x) := p\Big(\frac{\eta z}{\varepsilon\theta}, x\Big)e^{-i\frac{\eta k_0 z}{\varepsilon\theta^2}} \qquad \Big(\Big(\frac{\varepsilon\theta}{\eta}\Big)^2\partial_z^2 + \frac{k_0^2}{\theta^2}\Big)e^{i\frac{\eta k_0 z}{\varepsilon\theta^2}} = 0$$

solves

$$\Big[\Big(\frac{\varepsilon\theta}{\eta}\Big)^2\partial_z^2 + 2ik_0\partial_z + \frac{\eta}{\varepsilon}\Delta_x + \frac{k_0^2}{(\eta\varepsilon\theta)^{\frac{1}{2}}}\nu\Big(\frac{\eta z}{\varepsilon\theta}, x\Big)\Big]u = 0.$$

Paraxial Model in weak coupling regime



$$\Big(\Big(\frac{\varepsilon\theta}{\eta}\Big)^2\partial_z^2 + 2ik_0\partial_z + \frac{\eta}{\varepsilon}\Delta_x + \frac{k_0^2}{(\eta\varepsilon\theta)^{\frac{1}{2}}}\nu\Big(\frac{\eta z}{\varepsilon\theta}, x\Big)\Big)u = 0$$

Assuming $\left(\frac{\varepsilon\theta}{\eta}\right)^2\partial_z^2 u$ *negligible*, we obtain the paraxial model

$$\boxed{\Big(2ik_0\partial_z + \frac{\eta}{\varepsilon}\Delta_x + \frac{k_0^2}{(\eta\varepsilon\theta)^{\frac{1}{2}}}\nu\Big(\frac{\eta z}{\varepsilon\theta}, x\Big)\Big)u^\theta = 0} \qquad \boxed{u^\theta(0,x) = u_0(\varepsilon x)}.$$

- This amounts to *neglecting* backscattering.
- Difficult to justify. $d = 1$ [Bailly Clouet Fouque 96]; [Garnier Sølna 09]
- Assume $\varepsilon = \varepsilon(\theta)$ and $\eta = \eta(\theta)$ to simplify.

Itô-Schrödinger approximation

Paraxial model

$$\Big(2ik_0\partial_z + \frac{\eta}{\varepsilon}\Delta_x + \frac{k_0^2}{(\eta\varepsilon\theta)^{\frac{1}{2}}}\nu\Big(\frac{\eta z}{\varepsilon\theta}, x\Big)\Big)u^\theta = 0.$$

Formally, from central limit scaling, as $\tau \to 0$,

$$\frac{1}{\sqrt{\tau}}\nu(\frac{z}{\tau}, x)dz \approx dB(z, x).$$

As $\theta \to 0$, Paraxial approximated by Stratonovich-Schrödinger

$$du^\varepsilon = \frac{i\eta}{2k_0\varepsilon}\Delta_x u^\varepsilon dz + \frac{ik_0}{2\eta}u^\varepsilon \circ dB, \quad u^\varepsilon(0, x) = u_0(\varepsilon x),$$

and after Stratonovich correction by Itô-Schrödinger SPDE model

$$\boxed{du^\varepsilon = \frac{i\eta}{2k_0\varepsilon}\Delta_x u^\varepsilon dz - \frac{k_0^2 R(0)}{8\eta^2}u^\varepsilon dz + \frac{ik_0}{2\eta}u^\varepsilon dB}, \quad u^\varepsilon(0, x) = u_0(\varepsilon x).$$

[Dawson Papanicolaou 84] [Fannjiang Sølna 04]. *Simply false when* $\theta = 1$.

Paraxial & IS models. Main assumptions.

Paraxial model

$$\Big(2ik_0\partial_z + \frac{\eta}{\varepsilon}\Delta_x + \frac{k_0^2}{(\eta\varepsilon\theta)^{\frac{1}{2}}}\nu\Big(\frac{\eta z}{\varepsilon\theta}, x\Big)\Big)u^\theta = 0, \quad u^\theta(0,x) = u_0(\varepsilon x).$$

Itô-Schrödinger SPDE model

$$du^\varepsilon = \frac{i\eta}{2k_0\varepsilon}\Delta_x u^\varepsilon dz - \frac{k_0^2 R(0)}{8\eta^2}u^\varepsilon dz + \frac{ik_0}{2\eta}u^\varepsilon dB, \quad u^\varepsilon(0,x) = u_0(\varepsilon x).$$

- Random potential assumed stationary mean zero Gaussian

$$\mathbb{E}[\nu(z,x)\nu(z',x')] = \mathfrak{C}(z-z', x-x') \qquad \text{Short Range}$$

$$\mathbb{E}[B(z,x)B(z',x')] = \min(z,z')R(x-x'), \qquad R(x) = \int_{\mathbb{R}}\mathfrak{C}(s,x)ds$$

$\hat{R}(k) = \mathcal{F}R(k)$ sufficiently integrable with $Q = \nabla^2 R(0)$ *negative definite.*

Main Results.

Let u^θ and u^ε be solutions of Paraxial and IS models.

Assume incident profile $u_0(\varepsilon^\beta x)$ for $\beta \geq 1$. ($\beta > 1 \approx$ plane wave)

Assume $\varepsilon = \varepsilon(\theta)$ and $\eta = \eta(\theta)$ with $\varepsilon^{N\gamma} < \theta < \varepsilon^\gamma$ for $\gamma > 0$.

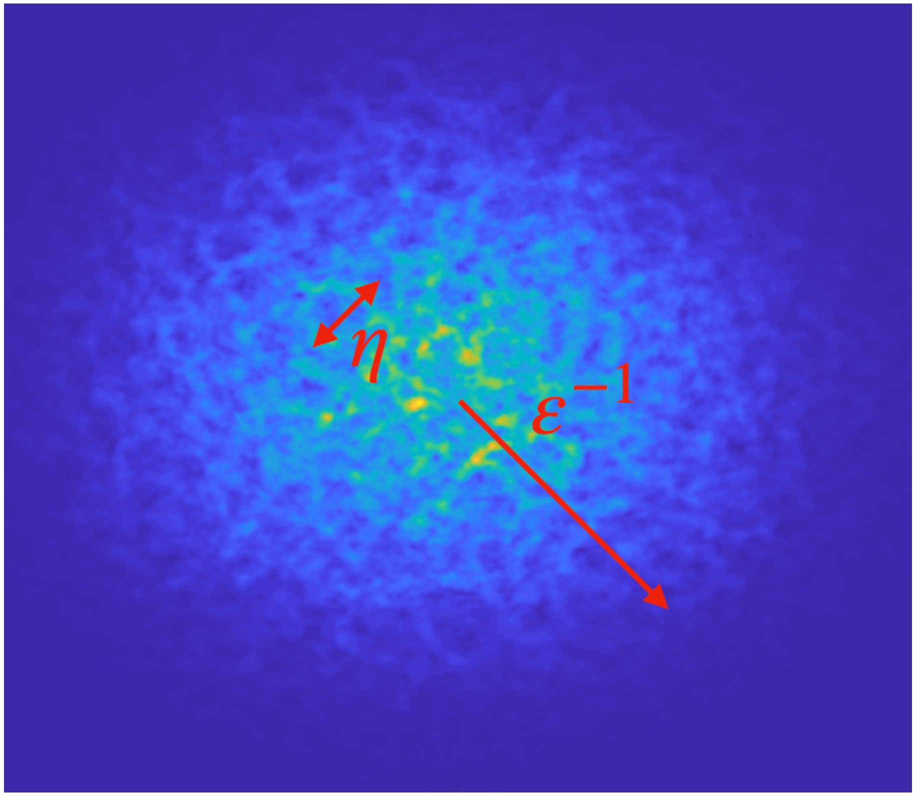
Let $\eta = 1$ in *kinetic* regime and $\eta \approx (\ln\ln\varepsilon^{-1})^{-1} \to 0$ in *diffusive* regime (assumed from now on).

For (z, r) *fixed*, define *rescaled* random field:

$$\boxed{x \mapsto \phi^\theta(z, r, x) := u^\theta\Big(z, \frac{r}{\varepsilon^\beta} + \eta x\Big).}$$

Theorem. [Gaussian conjecture] As $\theta \to 0$, and at *fixed* $z > 0$ and *fixed* $r \in \mathbb{R}^d$, $x \mapsto \phi^\theta$ and $x \mapsto \phi^\varepsilon$ converge in law to (the same) complex circular Gaussian field (with appropriate modifications in kinetic regime).

Speckle Scaling



- Let $z > 0$ and r/ε^β fixed. ($\beta = 1$ in above picture.)
- Process $x \mapsto \phi^\theta(x) = u^\theta(z, \frac{r}{\varepsilon^\beta} + \eta x)$ complex circular Gaussian as $\theta \to 0$.
- x parametrizes scale of correlation length. Regime $\lambda \ll \eta x \ll l_0$.

Limiting Process and Speckle scaling (plane wave)

In diffusive regime, Optical Length is $L = \frac{z}{\eta^2}$.

Let $\beta > 1$ with $u^\theta(0, z) = u_0(\varepsilon^\beta x)$. (very wide incident profile)

At (z, r) fixed, random field

$$x \mapsto \phi^\theta(x) = u^\theta(z, \frac{r}{\varepsilon^\beta} + \eta x)$$

converges to mean-zero complex Gaussian field $\phi(x)$ characterized by correlation function

$$\begin{aligned} \mathbb{E}\{\phi(x)\phi^*(y)\} &= |u_0(r)|^2 e^{-Cz|x-y|^2}, \qquad C = |\nabla^2 R(0)|\frac{k_0^2}{32} \\ \mathbb{E}\{\phi(x)\phi(y)\} &= 0. \end{aligned}$$

Correlation length of u^θ, whence scale of speckle, is $(LC)^{-\frac{1}{2}} = \frac{\eta}{\sqrt{Cz}}$.

Limit $z \to \infty$ of u^θ is singular. [Fouque Papanicolaou Samuelides 98].

Validity $\lambda \ll \frac{\eta}{\sqrt{Cz}} \ll l_0$.

Limiting Process and Speckle scaling when $\beta = 1$

Let $\beta = 1$ with $u^\theta(0,x) = u_0(\varepsilon x)$ incident beam.

Then $x \mapsto \phi^\theta(x) = u^\theta(z, \frac{r}{\varepsilon} + \eta x)$ converges to mean-zero complex Gaussian field $\phi(x)$ characterized by $\mathbb{E}\{\phi(x)\phi(y)\} = 0$ and correlation function $\mathcal{C}(x,y) = \mathbb{E}\{\phi(x)\phi^*(y)\}$

$$\mathcal{C}(x,y) \;=\; e^{\frac{k_0^2}{32} z (y-x)^t \Gamma (y-x)} e^{-i\frac{3k_0}{2z}(y-x)\cdot r} \; G(z^3, r) * [e^{i\frac{3k_0}{2z}(y-x)\cdot r} |u_0|^2(r)]$$

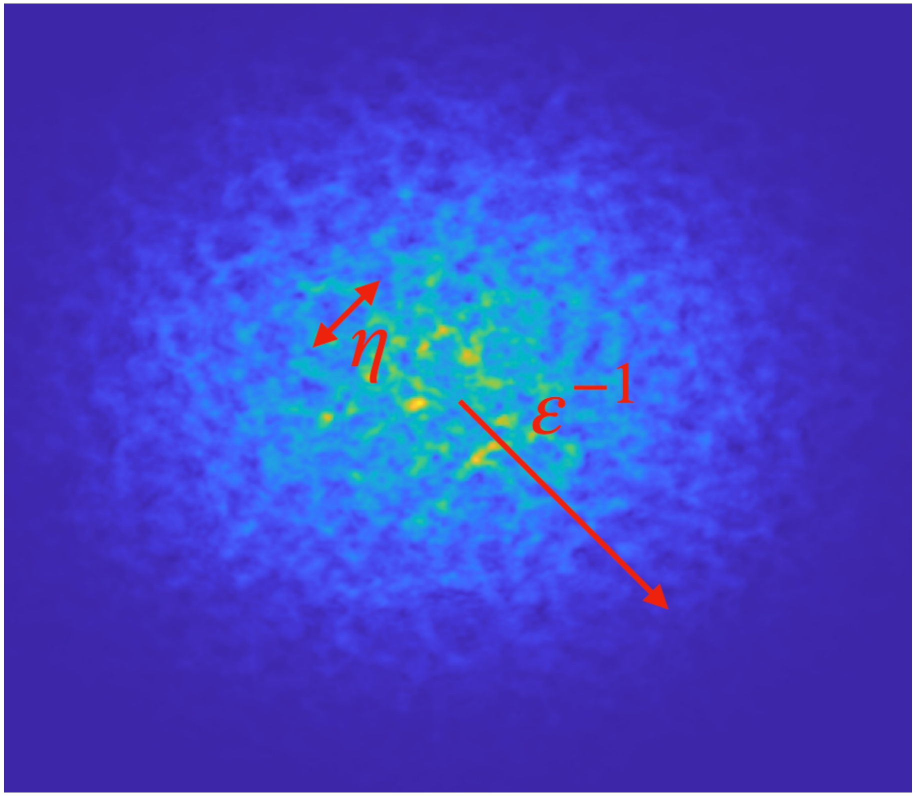
where $\Gamma = \nabla^2 R(0)$ (negative definite) and G Green's function of

$$(\partial_t + \frac{1}{24}\nabla_r \cdot \Gamma \nabla_r) G(t,r) = 0, \quad G(0,r) = \delta_0(r).$$

Anomalous Diffusion $t \equiv z^3 \approx \langle x^2 \rangle$ reflecting beam dispersion.

Correlation length of u^θ and scale of speckle $L^{-\frac{1}{2}} = \frac{\eta}{\sqrt{Cz}}$.

Speckle Scaling



- Process $x \mapsto \phi^\theta(x) = u^\theta(z, \frac{r}{\varepsilon} + \eta x)$ circular Gaussian as $\theta \to 0$
- (z, r) parametrize large-scale anomalous diffusion envelope
- x parametrizes scale of correlation length

Intensity in diffusive regime

Let $\beta = 1$. Define $I^\theta(z,r,x) = |\phi^\theta(z,r,x)|^2$.

• *Distribution of energy*. R.v. I^θ converges to I in distribution with I exponentially distributed with $\mathbb{E}I = I_2(z,r)$ and

$$\partial_{z^3} I_2 + \frac{1}{24}\nabla_r \cdot \Gamma \nabla_r I_2 = 0, \qquad I_2(0,r) = |u_0(r)|^2.$$

• Scintillation index in diffusive regime is (asymptotically) unity

$$\mathsf{S}^\theta(z,r,x) = \frac{\mathbb{E}[I^\theta(z,r,x)^2] - \mathbb{E}[I^\theta(z,r,x)]^2}{\mathbb{E}[I^\theta(z,r,x)]^2} \to 1$$

• Intensity correlation. For field $I(x) = |\phi(x)|^2$, we have

$$\mathbb{E}I(x)I(y) - \mathbb{E}I(x)\mathbb{E}I(y) = \mathbb{E}\phi(x)\phi^*(y)\mathbb{E}\phi(y)\phi^*(x) = |\mathcal{C}(x,y)|^2$$

• Self-averaging. For D centered cube of length $1 \gg a_\varepsilon \gg \varepsilon\eta$:

$$\frac{1}{|D|}\int_D I^\theta(z, r+r', x)\mathrm{d}r' \ \Rightarrow\ \mathbb{E}I(z,r) = I_2(z,r).$$

Convergences

• Finite dimensional distributions. Let $X = (x_1, \ldots, x_N)$ and

$$\Phi^\theta(z,r,X) = (\phi^\theta(z,r,x_1), \cdots, \phi^\theta(z,r,x_N)) = (u^\theta(z, \varepsilon^{-\beta} r + \eta x_1), \cdots).$$

We prove that random vector $\Phi^\theta \Rightarrow \Phi$ with Φ circularly symmetric Gaussian random vector with elements $\{\phi_j\}_{j=1}^N$ s.t.

$$\mathbb{E}[\phi_j \phi_l] = 0, \quad \mathbb{E}[\phi_j \phi_l^*] = \mathcal{C}(x_j, x_l)\,.$$

• Stochastic continuity & tightness. We prove for some $\alpha_0 > 0$

$$\sup_{s\in[0,z]} \mathbb{E}|\phi^\theta(s,r,x+h) - \phi^\theta(s,r,x)|^{2n} \le C(z,\alpha_0)|h|^{2\alpha_0 n}, \;\; h \in B(0,1) \subset \mathbb{R}^d.$$

• This shows tightness of $\mathbb{P}_\theta$ generated by $x \mapsto \phi^\theta(x)$ on (Hölder) continuous functions and convergence to $\mathbb{P}$ the law of $x \mapsto \phi(x)$.

Derivation I

- Main difficulty is proving convergence for IS $u^\varepsilon(\cdot)$.
- Statistical moments of paraxial model $u^\theta(\cdot)$ satisfy these equations *approximately* (Duhamel expansion). Extends results from IS to paraxial.
- Statistical moments of IS model satisfy closed form equations.

- Focus on IS model $u^\varepsilon(\cdot)$. Moments and closed form equations are:

$$\mu^\varepsilon_{p,q}(z,X,Y) = \mathbb{E}\Big[\prod_{j=1}^p u^\varepsilon(z,x_j)\prod_{l=1}^q u^{\varepsilon *}(z,y_l)\Big],$$

$$\partial_z \mu^\varepsilon_{p,q} = \mathcal{L}^\varepsilon_{p,q}\mu^\varepsilon_{p,q}, \qquad \mathcal{L}^\varepsilon_{p,q} := \frac{i\eta}{2k_0\varepsilon}\Big(\sum_{j=1}^p \Delta_{x_j} - \sum_{l=1}^q \Delta_{y_j}\Big) + \frac{k_0^2}{4\eta^2}\mathcal{U}_{p,q},$$

$$\mathcal{U}_{p,q} = \sum_{j=1}^p\sum_{l=1}^q R(x_j-y_l) - \sum_{1\le j<j'\le p} R(x_j - x_{j'}) - \sum_{1\le l<l'\le q} R(y_l - y_{l'}) - \frac{p+q}{2}R(0).$$

Derivation II

(X, Y) with dual variables v. Then phase-compensated moments

$$\psi^{\varepsilon}_{p,q}(z,v) = e^{-\frac{iz\eta}{2k_0\varepsilon} v^{\mathrm{t}}\Theta v} \widehat{\mu}^{\varepsilon}_{p,q}(z,v) \qquad \text{solves}$$

$$(\partial_z - L^{\varepsilon}_{p,q})\psi^{\varepsilon}_{p,q} = 0, \quad L^{\varepsilon}_{p,q} = \sum_j L^{\varepsilon}_j \qquad \text{with}$$

$$L^{\varepsilon}_j \rho(z,v) = \frac{c_j}{\eta^2} \int_{\mathbb{R}^d} \widehat{R}(k) e^{\frac{iz\eta}{2k_0\varepsilon}(v,k)^{\mathrm{t}} \breve{A}_j(v,k)} \rho(z, v - A_j k) dk.$$

Define *solution operator* $\psi^{\varepsilon}_{p,q}(z) = U^{\varepsilon}_{p,q}\psi^{\varepsilon}_{p,q}(0)$.

Theorem. $U^{\varepsilon}_{p,q} = N^{\varepsilon}_{p,q} + E^{\varepsilon}_{p,q}$ where $N^{\varepsilon}_{p,q}$ corresponds to the (p,q) moment of a circular complex Gaussian field and where

$$\|E^{\varepsilon}_{p,q}\| \ll c(p,q,z)\varepsilon^{\frac{1}{3}},$$

for the choice $\eta^{-1} \approx \ln\ln \varepsilon^{-1}$ and $c(p,q,z)$ bounded on compact domains.

- $\|\cdot\|$ is TV norm on $\mathcal{M}_B(\mathbb{R}^{pd+qd})$ or corresponding operator norm.

Derivation III

Advantage of the Banach space $\mathcal{M}_B(\mathbb{R}^{pd+qd})$ is that error term $E^{\varepsilon}_{p,q}$ translates into a controlled error in the uniform sense in the physical variables.

Thus,

$$\mu^{\varepsilon}_{p,q}(z,X,Y) = \mathbb{F}\Big(\mu^{\varepsilon}_{1,1}(z,x_1,y_1),\cdots,\mu^{\varepsilon}_{1,1}(z,x_p,y_q)\Big) + \mathcal{O}_{\|\cdot\|_\infty}(\varepsilon^{\frac{1}{3}}c(p,q,z))$$

where $\mathbb{F}$ is continuous functional describing (p,q) moments of mean zero complex circular Gaussian variable in terms of its second moments.

(To be modified in kinetic regime where first moments do not vanish).

This (plus tightness of random vector) characterizes limit

$$(\phi^{\varepsilon}(z,r,x_1),\ldots) = \Phi^{\varepsilon}(z,r,X) \Rightarrow \Phi.$$

Φ circularly Gaussian r.v. Fully characterized by its moments (Carleman criterion) and hence unique.

Derivation IV

$$\mu^\varepsilon_{p,q}(z,X,Y) = \mathbb{F}(\mu^\varepsilon_{1,1}(z,x_1,y_1),\cdots,\mu^\varepsilon_{1,1}(z,x_p,y_q)) + \mathcal{O}_{\|\cdot\|_\infty}(\varepsilon^{\frac{1}{3}}c(p,q,z))$$

shows that moments characterized by (limit as $\varepsilon \to 0$) second moments $\mu_{1,1}(z,x,y)$ given by

$$\partial_z \mu^\varepsilon_{1,1} = \frac{i\eta}{2k_0\varepsilon}(\Delta_x - \Delta_y)\mu^\varepsilon_{1,1} + \frac{k_0^2}{4\eta^2}\Big(R(x-y) - R(0)\Big)\mu^\varepsilon_{1,1}.$$

Direct analysis gives limit $\varepsilon \to 0$ for $\mathcal{C}(x,y) = \mathbb{E}\{\phi(x)\phi^*(y)\}$:

$$\mathcal{C}(x,y) \;=\; e^{\frac{k_0^2}{32}z(y-x)^t\Gamma(y-x)} e^{-i\frac{3k_0}{2z}(y-x)\cdot r}\; G(z^3,r) * [e^{i\frac{3k_0}{2z}(y-x)\cdot r}|u_0|^2(r)]$$

- Tightness of field $x \mapsto \phi^\varepsilon(z,r,x)$ obtained by proving

$$\sup_{s\in[0,z]} \mathbb{E}|\phi^\varepsilon(s,r,x+h) - \phi^\varepsilon(s,r,x)|^{2n} \le C(z,\alpha_0)|h|^{2\alpha_0 n},$$

also using closed form equations for moments. $\square$

Related work

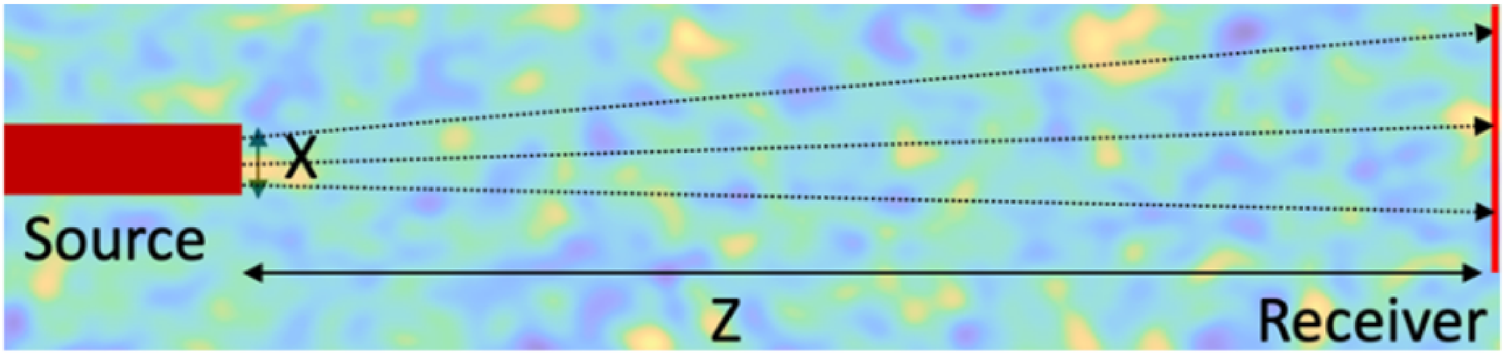
In [B. Komorowski Ryzhik ARMA 11], $\widehat{\phi} = \widehat{u}^{\varepsilon}(z,\xi)e^{\frac{i}{\varepsilon k_0}z\xi^2}$ in paraxial regime analyzed by *diagrammatic expansion*. Shows that r.v. $\widehat{\phi} - \mathbb{E}\widehat{\phi}$ complex Gaussian r.v. Also analyzes cases where ν is *long range*.

In [Gu Komorowski CPDE 21], result improved in Itô-Schrödinger regime. Shows that $(z,\zeta) \to \widehat{\phi}(z,\zeta) = \widehat{u}^{\varepsilon}(z,\xi)e^{\frac{i}{\varepsilon k_0}z\xi^2}$ converges in distribution on continuous functions to limit $\widehat{\phi}$ such that $\widehat{\phi} - \mathbb{E}\widehat{\phi}$ complex Gaussian field. Uses *martingale techniques*.

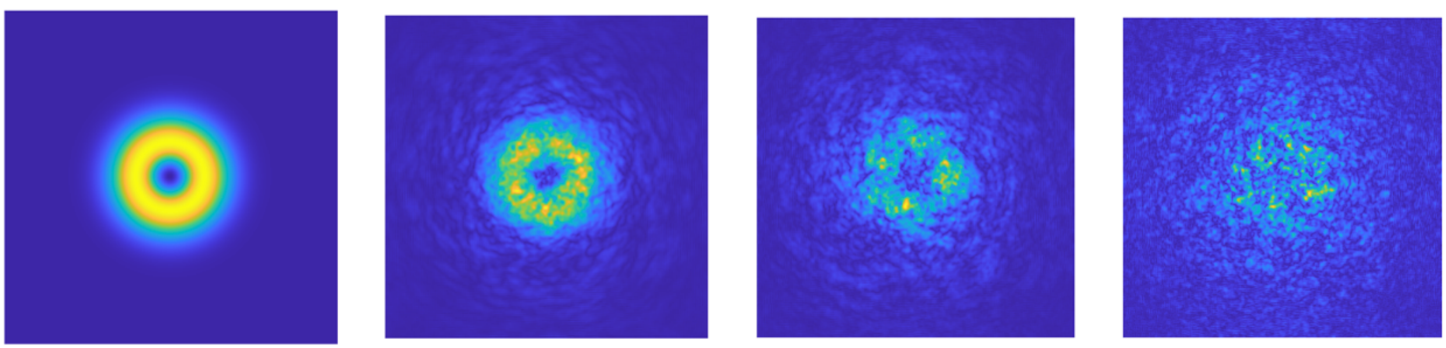
In [Garnier Sølna ARMA 16 & MMS 23], the *fourth moment of IS model* is analyzed in detail in the kinetic regime. Analysis shows that fourth moment is consistent with Gaussian conjecture. Show that scintillation index asymptotically 1 for large distances of propagation: $\mathbb{E}I \approx \mathbb{E}I^2$.

[Carminati Schotland 21] Scaling of speckle spot in $(LC)^{-\frac{1}{2}} = \frac{\eta}{\sqrt{Cz}}$ different from spot in diffusive regime through slab L, where $f\lambda \mathrm{L}^{-1}$.

Wave propagation in random media



Wave-beam propagating in turbulent atmosphere



Receiver reading as turbulence strength (or z) increases.

Splitting / phase screen Algorithms

• Phase screen algorithms routinely used to solve paraxial models [Schmidt 2010] [BPM Matlab 2021] even when smallest scale in problem $\theta \ll \Delta z$ in practice. No convergence guarantee.

• Splitting algorithm and full spatial discretization with at least *first-order* convergence guaranteed as $\Delta z \to 0$ for $\theta \in (0, 1]$ and *second-order* convergence for statistical moments.

B. Nair 2025. Splitting algorithms for paraxial and Itô-Schrödinger models of wave propagation in random media. arXiv:2503.00633

Paraxial & IS models

Paraxial model

$$\partial_z u^\theta = i\kappa_1 \Delta_x u^\theta + i\kappa_2 \frac{1}{\sqrt{\theta}} \nu\Big(\frac{z}{\theta}, x\Big) u^\theta, \quad u^\theta(0,x) = u_0(x).$$

Itô-Schrödinger SPDE model (essentially limit $\theta \to 0$)

$$du = i\kappa_1 \Delta_x u dz - \frac{\kappa_2^2 R(0)}{2} u dz + i\kappa_2 u dB, \quad u(0,x) = u_0(x).$$

- Random potential: stationary mean zero Gaussian

$$\mathbb{E}[\nu(z,x)\nu(z',x')] = \mathcal{C}(z-z', x-x') \qquad \text{Short Range}$$

$$\mathbb{E}[B(z,x)B(z',x')] = \min(z,z') R(x-x'), \qquad R(x) = \int_{\mathbb{R}} \mathcal{C}(s,x) ds$$

Splitting Algorithm

For $\gamma \in [0,1]$ with $\gamma = \frac{1}{2}$ centered (Strang) splitting:

$$\tau_\gamma(z) := \Delta z \sum_{n\geq 0} \delta(z - (n+\gamma)\Delta z).$$

$$\partial_z u^{\theta\Delta} = i\tau_\gamma(z)\kappa_1(z)\Delta_x u^{\theta\Delta} + i\kappa_2(z)\nu^\theta(z,x)u^{\theta\Delta}, \quad u^{\theta\Delta}(0,x) = u_0(x).$$

Splitting into succession of simple steps (Similar for SPDE)

$$u^{\theta\Delta}(z,x) = \begin{cases} \mathsf{V}^\theta_{z_n}(z)u^{\theta\Delta}(z_n,x), & z_n < z \leq z_n + \gamma\Delta z, \\ \mathsf{V}^\theta_{z_n+\gamma\Delta z}(z) \circ \mathcal{G}(\chi_{z_n}(z_{n+1})) \circ \mathsf{V}^\theta_{z_n}(z_n+\gamma\Delta z)u^{\theta\Delta}(z_n,x), \end{cases}$$

$$\mathsf{V}^\theta_{z_1}(z_2): \quad (\mathsf{V}^\theta\psi)(x) = \exp\Big(\int_{z_1}^{z_2} i\kappa_2(s)\nu^\theta(s,x)\mathrm{d}s\Big)\psi(x)$$

$$\mathcal{G}(z): \quad \mathcal{G}\psi(x) = \int_{\mathbb{R}^d} G(x-x',z)\psi(x')\mathrm{d}x', \qquad G(x,z) := \frac{1}{(4\pi i z)^{\frac{d}{2}}}e^{\frac{i|x|^2}{4z}}.$$

Locality in Fourier variables: $\mathcal{G}(t) = \mathcal{F}^{-1}e^{-it|\xi|^2}\mathcal{F}$; $\chi_s(t) = \int_s^t \kappa_1(z)dz$.

Convergence of Splitting Scheme

$\partial_z u^{\theta\Delta} = i\tau_\gamma(z)\kappa_1(z)\Delta_x u^{\theta\Delta} + i\kappa_2(z)\nu^\theta(z,x)u^{\theta\Delta}, \quad u^{\theta\Delta}(0,x) = u_0(x).$

Regime of interest: $\theta \ll \Delta z$. In practice, $\theta = \frac{\lambda}{\ell_{\text{corr}}} = 10^{-3}m$ for $L = 10^3 m$.

Splitting schemes typically *do not* converge (to right solution) when Δz not smallest scale. Standard commutator techniques $e^{h(A+B)} \approx e^{hA}e^{hB}$ with $[hA, hB]$ small do not apply directly for $A = \Delta_x$ and $B = \frac{1}{\sqrt{\theta}}\nu(\frac{z}{\theta}, x)$.

Here: as $\theta \to 0$, paraxial converges to SPDE model.

Splitting scheme converges for *both* paraxial and SPDE limit.

Pathwise convergence

Theorem. [Path-wise estimates] $|||u|||_{\mathbb{X}} := \sup_{0\le s\le Z}(\mathbb{E}\|u(s,\cdot)\|^2_{L^2(\mathbb{X})})^{\frac{1}{2}}$

1. Let $\mathbb{X} = \mathbb{R}^d$ and $\mathsf{v} \in \{u, u^\theta\}$. Then $|||\mathsf{v} - \mathsf{v}^\Delta|||_{\mathbb{X}} \le C\Delta z$.

2. Let $\mathbb{X} = \mathbb{R}^d$ and $\mathsf{v} \in \{u, u^\Delta, u^\theta, u^{\theta\Delta}\}$.
Then for $N \ge 1$, $|||\mathsf{v} - \mathsf{v}_c|||_{\mathbb{X}} \le C_N[(\Delta k)^2 + K_k^{-N}]$.

3. Let $\mathbb{X} = \mathbb{T}^d_L$ and $\mathsf{v} \in \{u, u^\Delta, u^\theta, u^{\theta\Delta}\}$.
Then for $N \ge 1$, $|||\mathsf{v}_c - \mathsf{v}_\sharp|||_{\mathbb{X}} \le C_N L^{-N}$.

4. Let $\mathbb{X} = \mathbb{T}^d_L$ and $\mathsf{v} \in \{u, u^\Delta, u^\theta, u^{\theta\Delta}\}$.
Then for $N \ge 1$, $|||\mathsf{v}_\sharp - \mathsf{v}_\delta|||_{\mathbb{X}} \le C_N(\Delta x)^N$.

Convergence of moments

Theorem. [Moment estimates] $D = [0, Z] \times \mathbb{X}^p \times \mathbb{X}^q$

1. Let $\mathsf{v} \in \{u, u^\theta\}$ and $\mathbb{X} = \mathbb{R}^d$. Then $\|\mu_{p,q}[\mathsf{v}] - \mu_{p,q}[\mathsf{v}^\Delta]\|_{L^\infty(D)} \le C(\Delta z)^\beta$.

2. Let $\mathsf{v} \in \{u, u^\Delta, u^\theta, u^{\theta\Delta}\}$ and $\mathbb{X} = \mathbb{R}^d$.
Then for $N \ge 1$, $\|\mu_{p,q}[\mathsf{v}] - \mu_{p,q}[\mathsf{v}_c]\|_{L^\infty(D)} \le C_N[(\Delta k)^2 + K_k^{-N}]$.

3. Let $\mathbb{X} = \mathbb{T}_L^d$ and $\mathsf{v} \in \{u, u^\Delta, u^\theta, u^{\theta\Delta}\}$.
Then for $N \ge 1$, $\|\mu_{p,q}[\mathsf{v}_c] - \mu_{p,q}[\mathsf{v}_\sharp]\|_{L^\infty(D)} \le C_N L^{-N}$.

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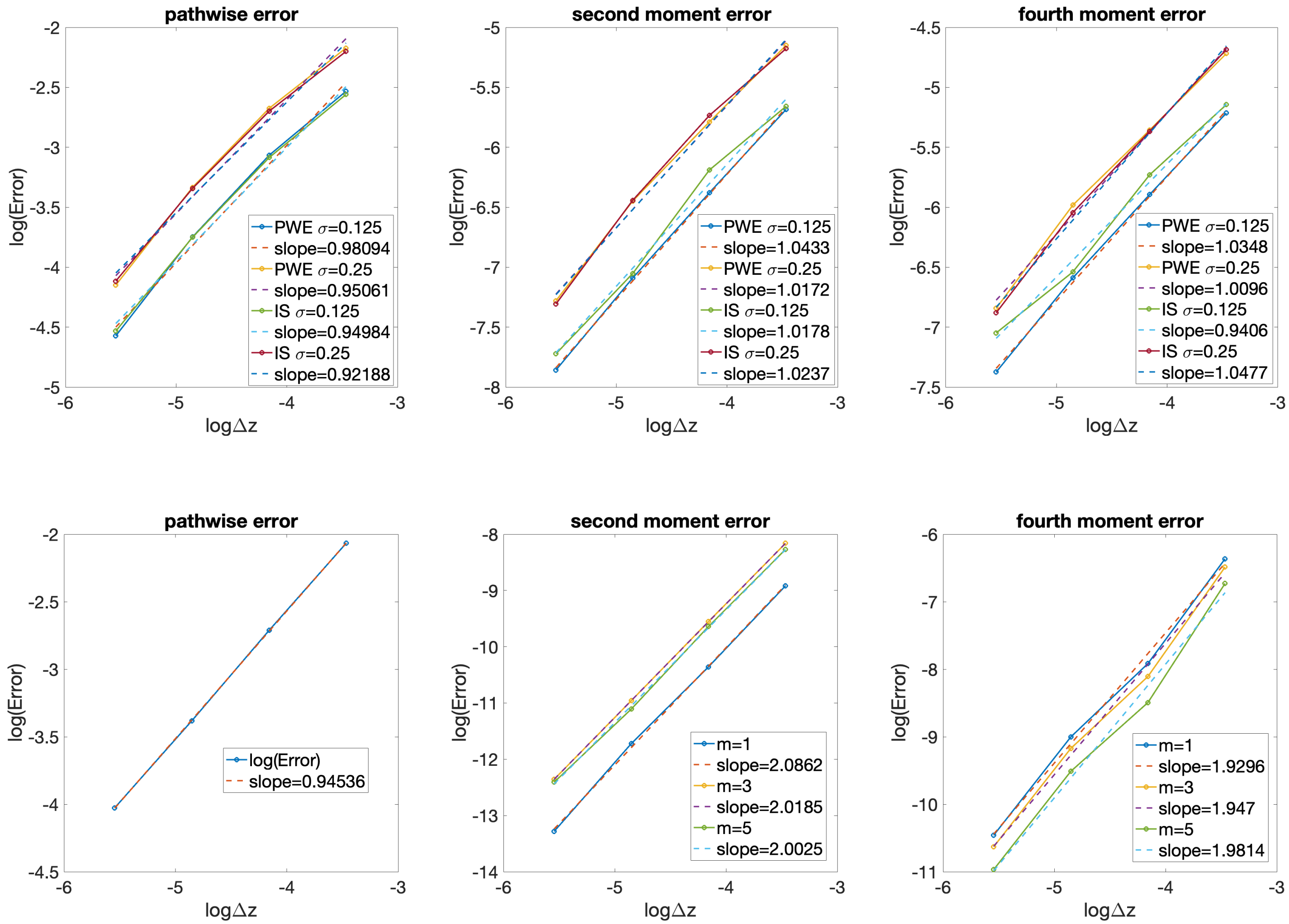
Then for $N \ge 1$, $\|\mu_{p,q}[\mathsf{v}_\sharp] - \mu_{p,q}[\mathsf{v}_\delta]\|_{L^\infty(D)} \le C_N(\Delta x)^N$.

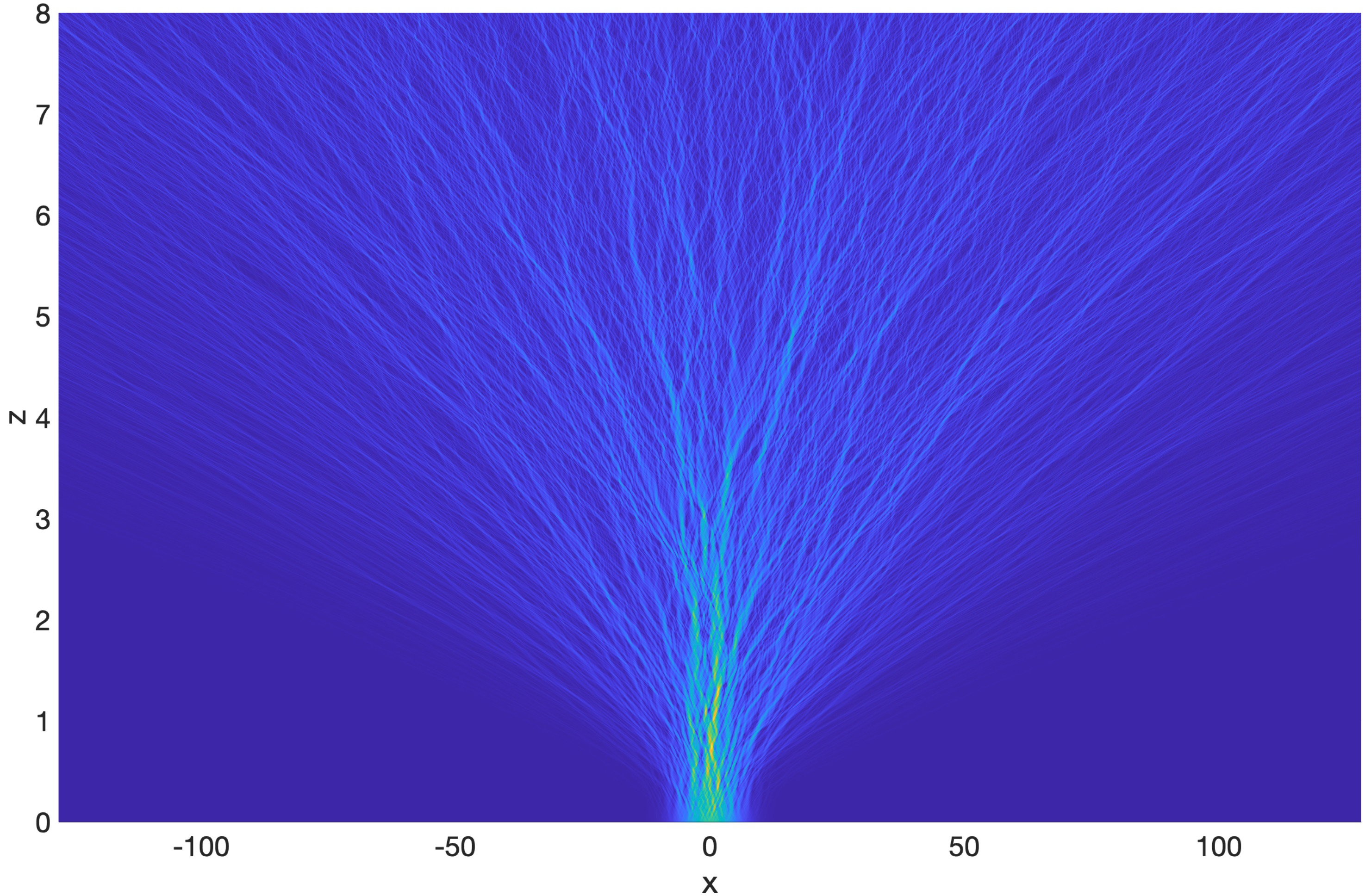
Above $\beta = 1$ when $\gamma \ne 12$ while $\beta = 2$ when $\gamma = \frac{1}{2}$ for SPDE and $\frac{3}{2} \le \beta \le 2$ for paraxial with $\beta = 2$ when $\theta \le \Delta z$ and $\beta = \frac{3}{2}$ when $\Delta z = \theta^2$.

Numerical simulations



Pathwise and moment errors for $\gamma = 1$ and $\gamma = \frac{1}{2}$ (10^8 realizations).

Long distance simulations



$(1+1)$d. Speckle spot much smaller than (algebraic) correlations in z

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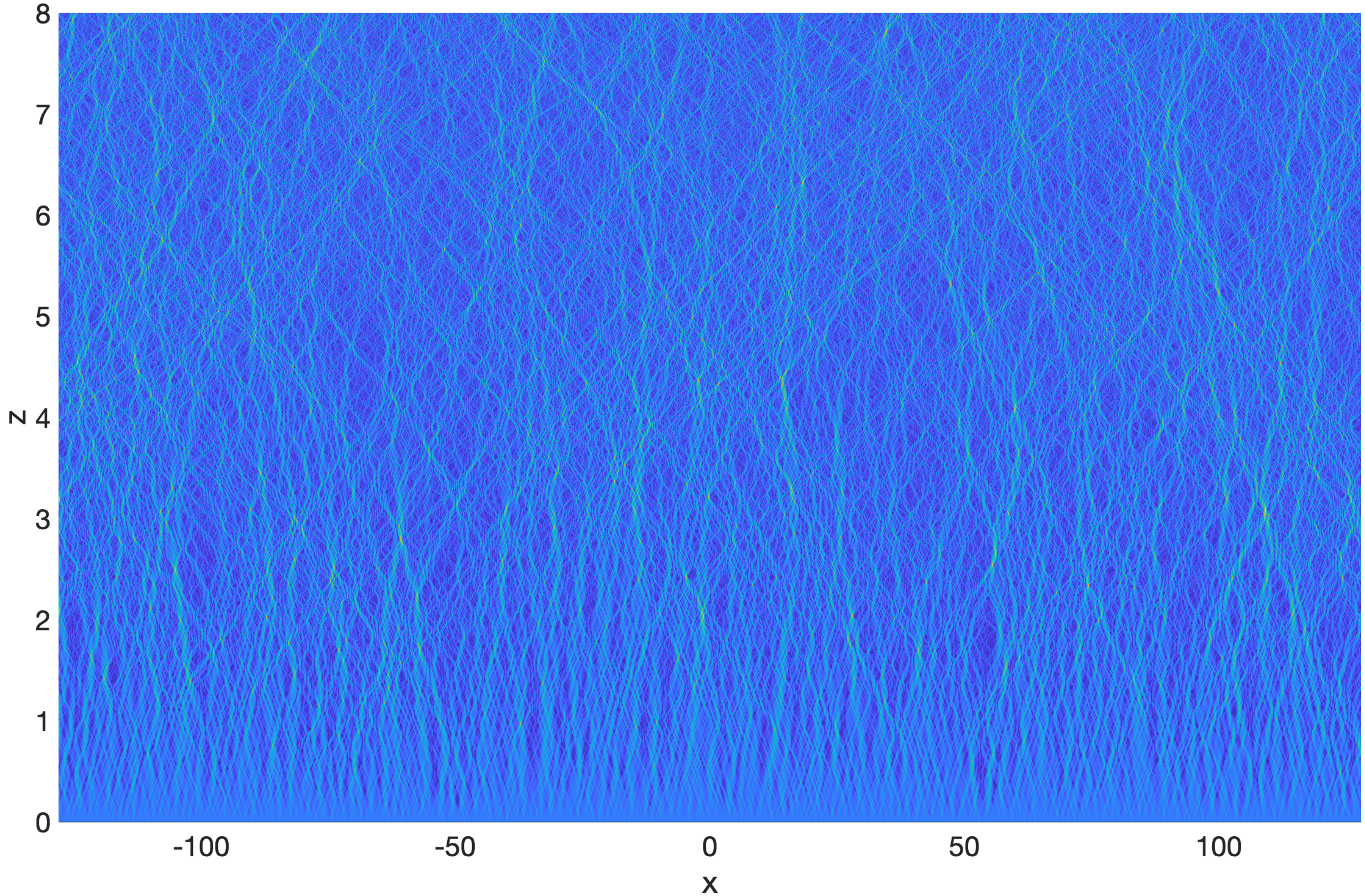

$(1+1)$d. Speckle spot much smaller than (algebraic) correlations in z

Summary

Theorem. As $\theta \to 0$, and at *fixed* $z > 0$ and *fixed* $r \in \mathbb{R}^d$, paraxial $x \mapsto \phi^\theta$ and IS $x \mapsto \phi^\varepsilon$ converge in law to a complex circular Gaussian field.

- Field characterized by $\mathcal{C}(x,y) = \mathbb{E}\{\phi(x)\phi^*(y)\}$ with

$$\mathcal{C}(x,y) \;=\; e^{\frac{k_0^2}{32}z(y-x)^t\Gamma(y-x)}e^{-i\frac{3k_0}{2z}(y-x)\cdot r}\; G(z^3,r) * [e^{i\frac{3k_0}{2z}(y-x)\cdot r}|u_0|^2(r)]$$

$$(\partial_t + \frac{1}{24}\nabla_r \cdot \Gamma\nabla_r)G(t,r) = 0, \quad G(0,r) = \delta_0(r).$$

Anomalous Diffusion $t = z^3 \approx \langle x^2 \rangle$ reflecting beam dispersion.

- B. Nair 2024. Complex Gaussianity of long-distance random wave processes. Arxiv arXiv:2402.17107
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Thank you!