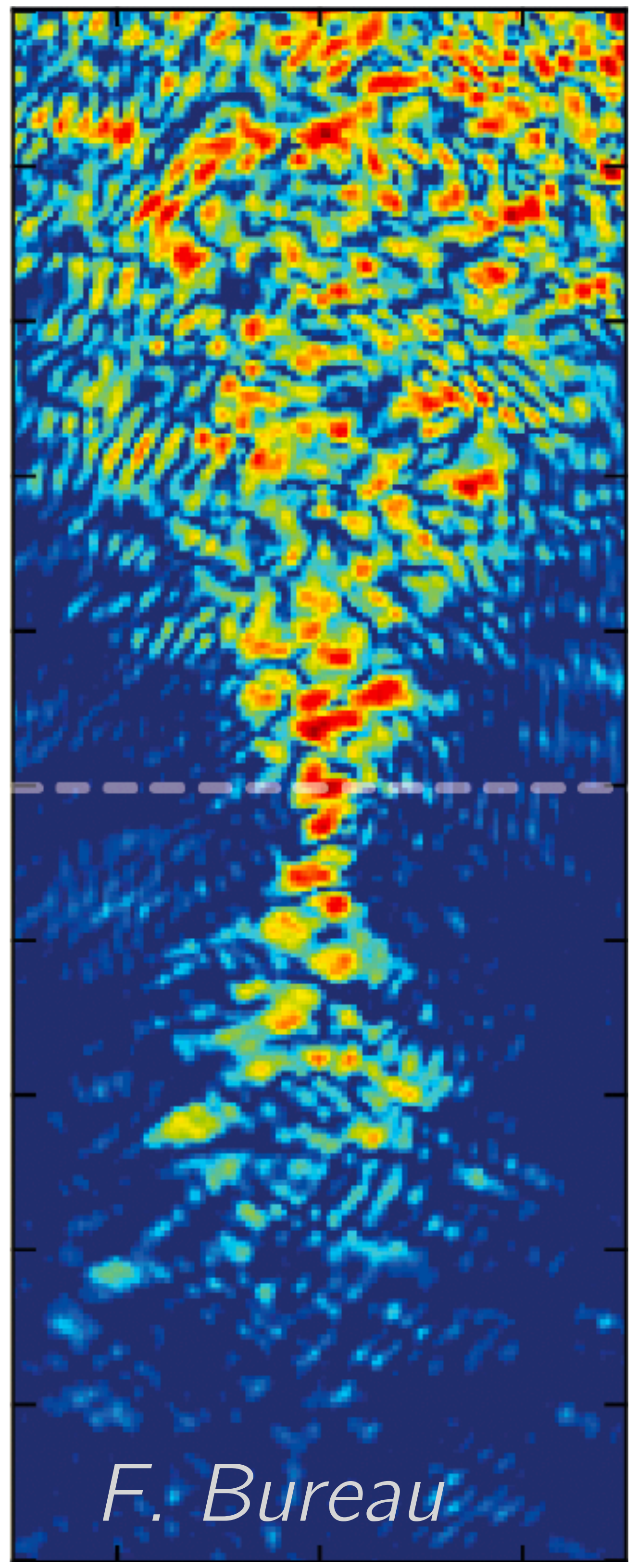
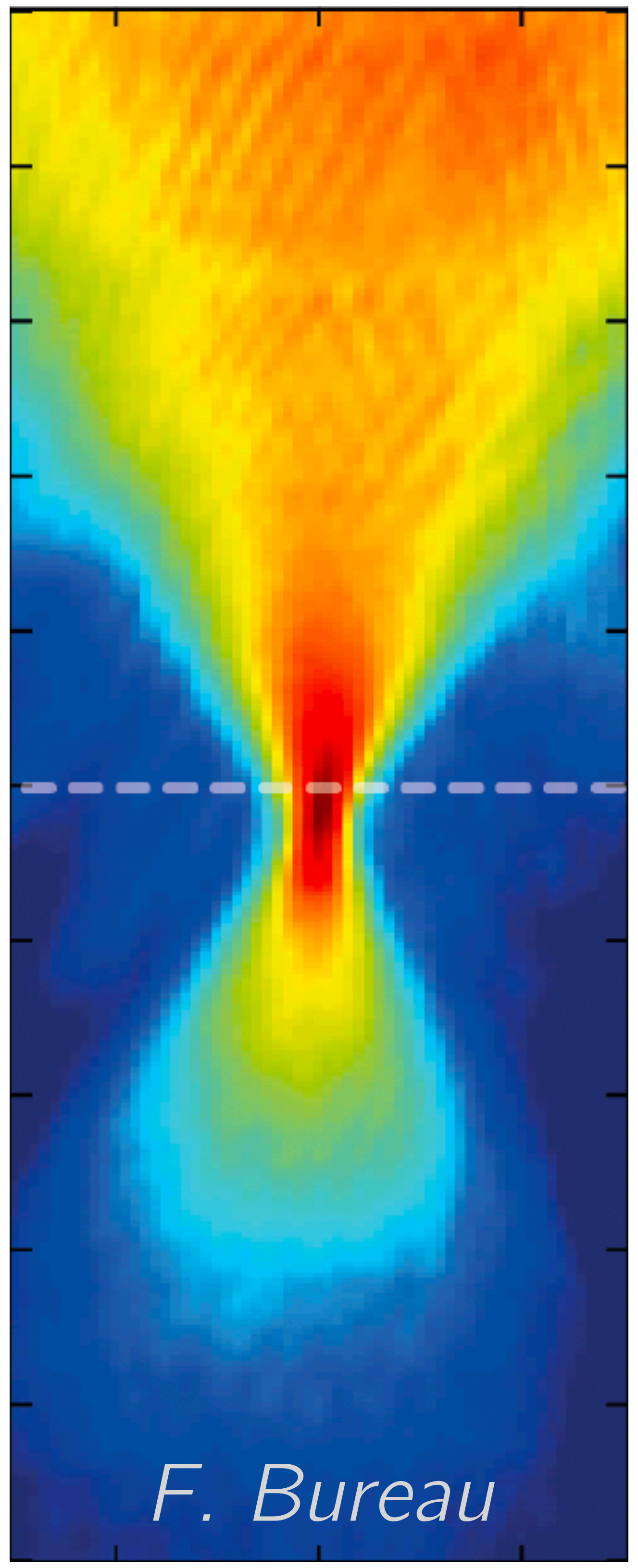


Self-Imaging Waves

Probing the speckle to estimate the effective speed of sound, a first step towards quantitative ultrasound imaging

Joint work w/ J. Garnier, L. Giovangigli, Q. Goepfert, E. Parolin



Institut **Langevin**
ONDES ET IMAGES ESPCI PARIS | PSL

Adaptative optics in 1 min

Theoretical model for a linear imaging system → Theoretical Point Spread Function (PSF)

Adaptative optics in 1 min

Theoretical model for a linear imaging system $\longrightarrow$ Theoretical Point Spread Function (PSF)

Practical situation with point-like object in field of view $\xrightarrow{\text{Fitting of the PSF}}$ Adjust perturbative parameters (aberrations, travel time...) to find the focusing law

Adaptative Optics

(Compensating modelisation approximations)

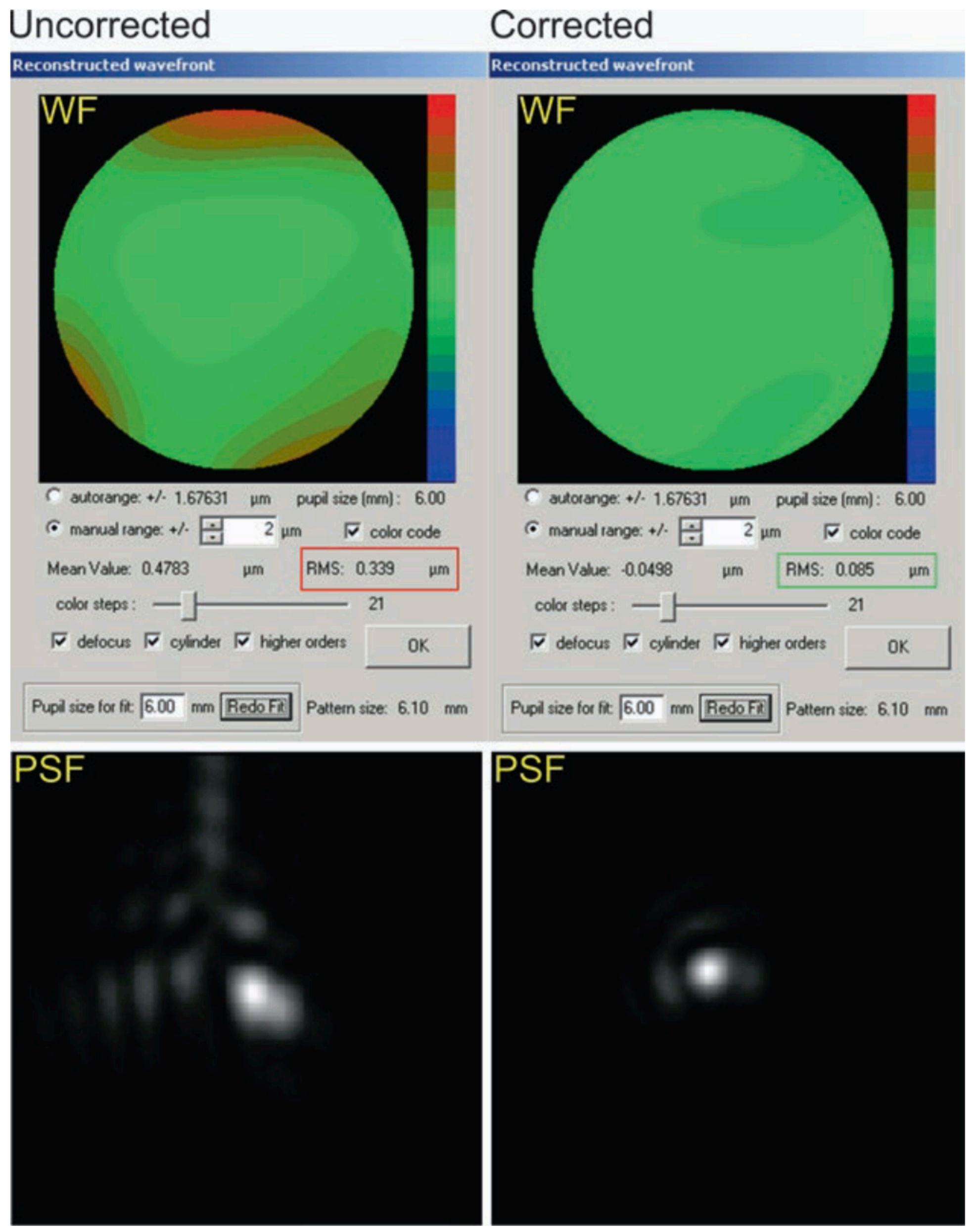
Adaptative optics in 1 min

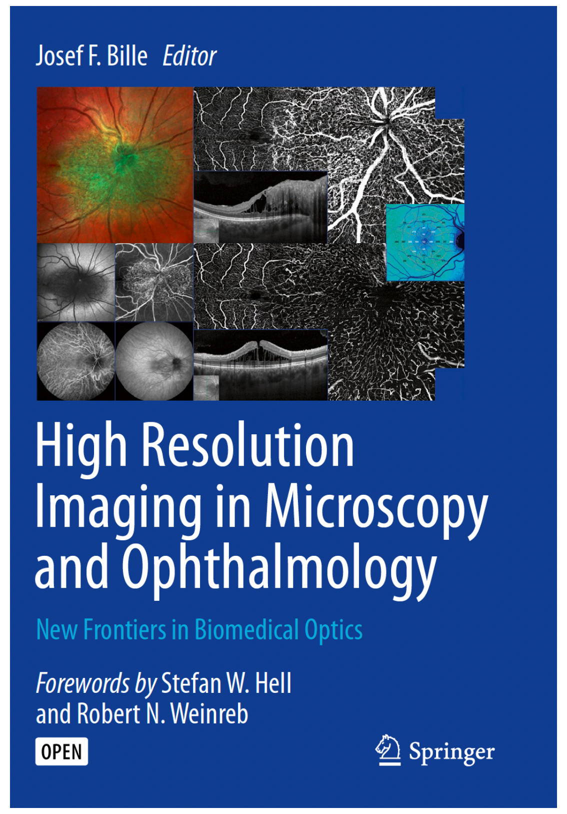
Theoretical model for a linear imaging system → Theoretical Point Spread Function (PSF)

Practical situation with point-like object in field of view $\xrightarrow{\text{Fitting of the PSF}}$ Adjust perturbative parameters (aberrations, travel time…) to find the focusing law

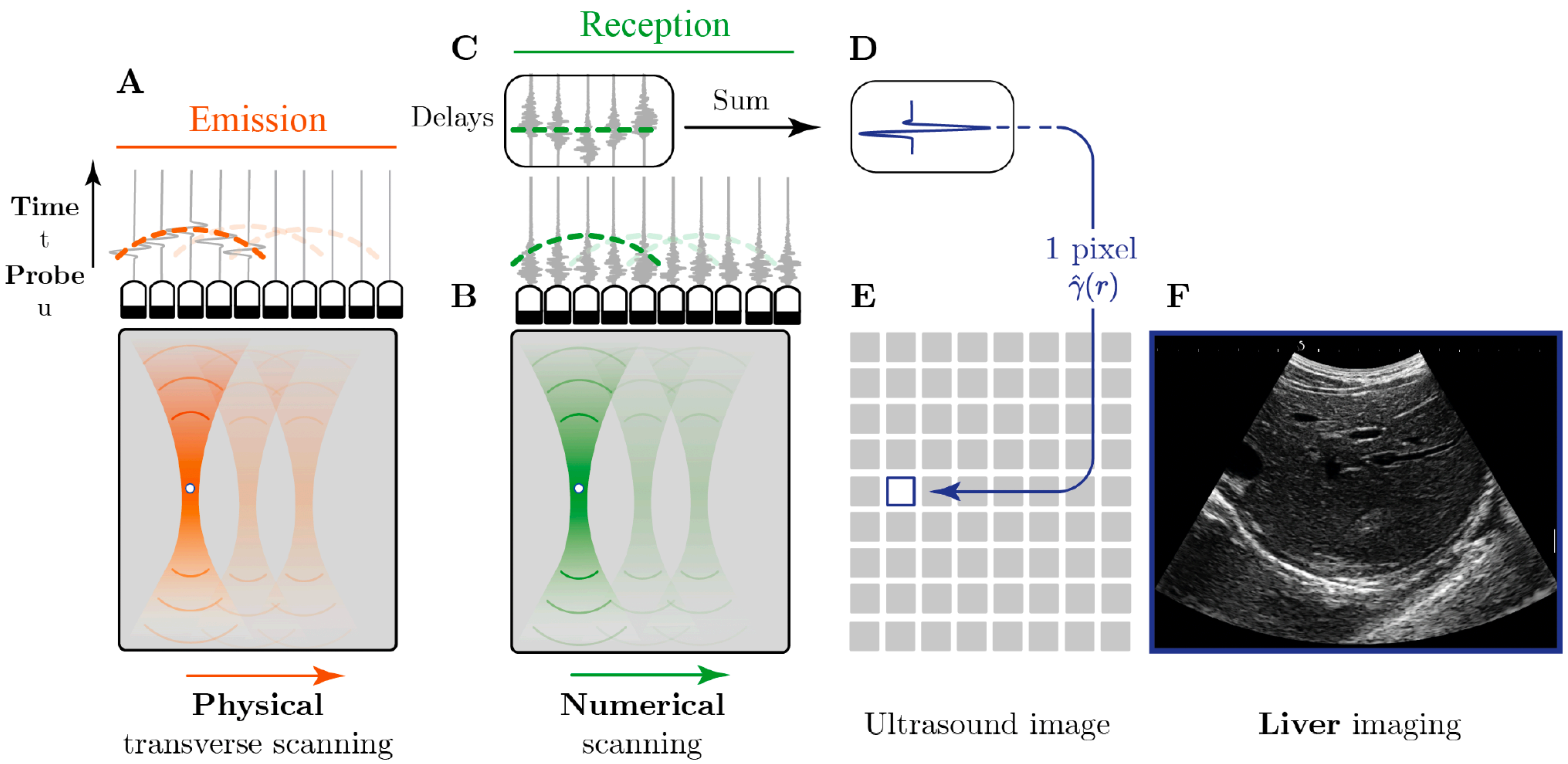
Adaptative Optics

(Compensating modelisation approximations)





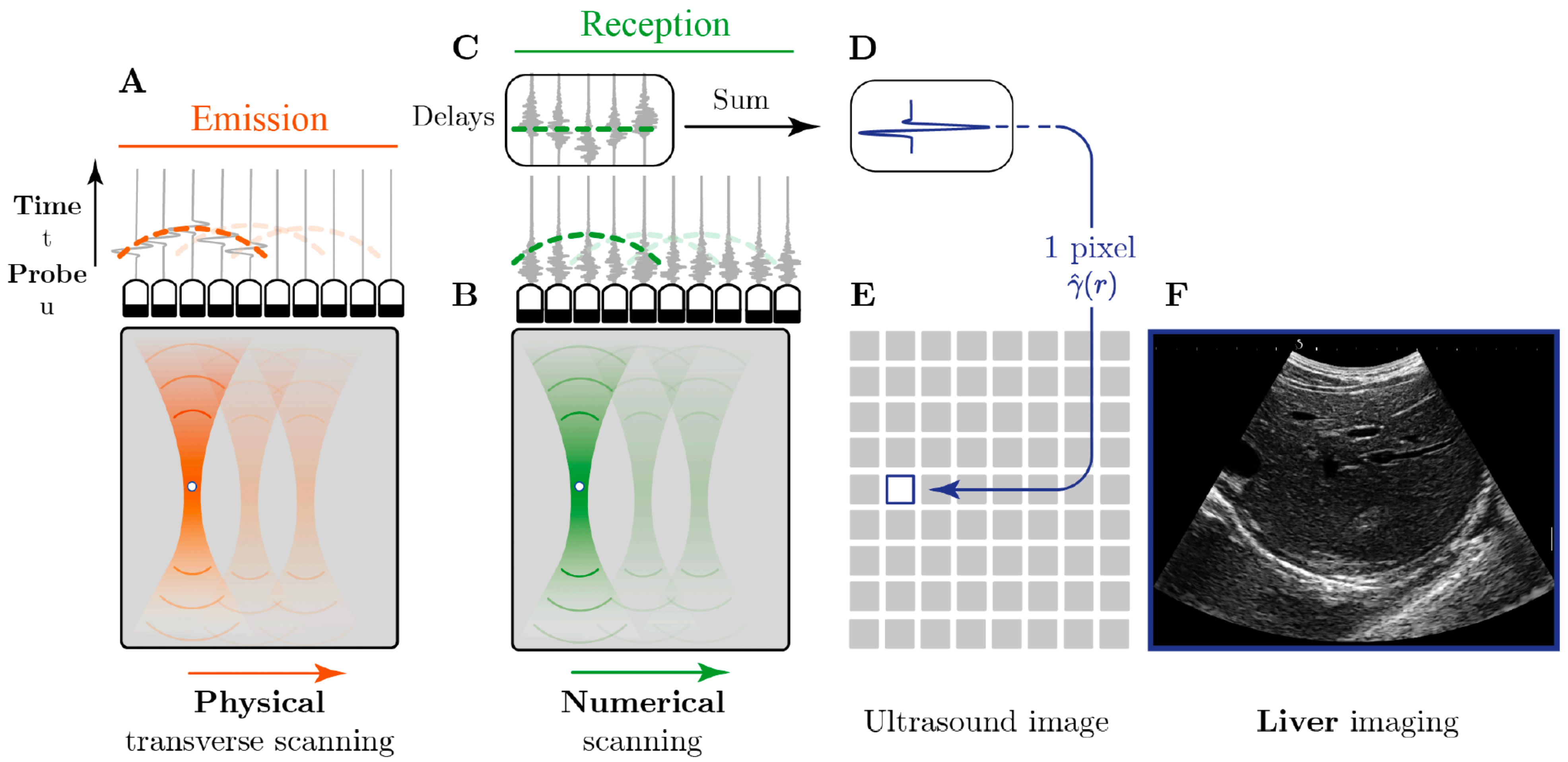
Principle of ultrasound imaging



Flavien Bureau: *Analyse multi-dimensionnelle de la matrice de réflexion pour l'imagerie ultrasonore quantitative*

Principle of ultrasound imaging

Image is computed using space time correspondance



Principle of ultrasound imaging

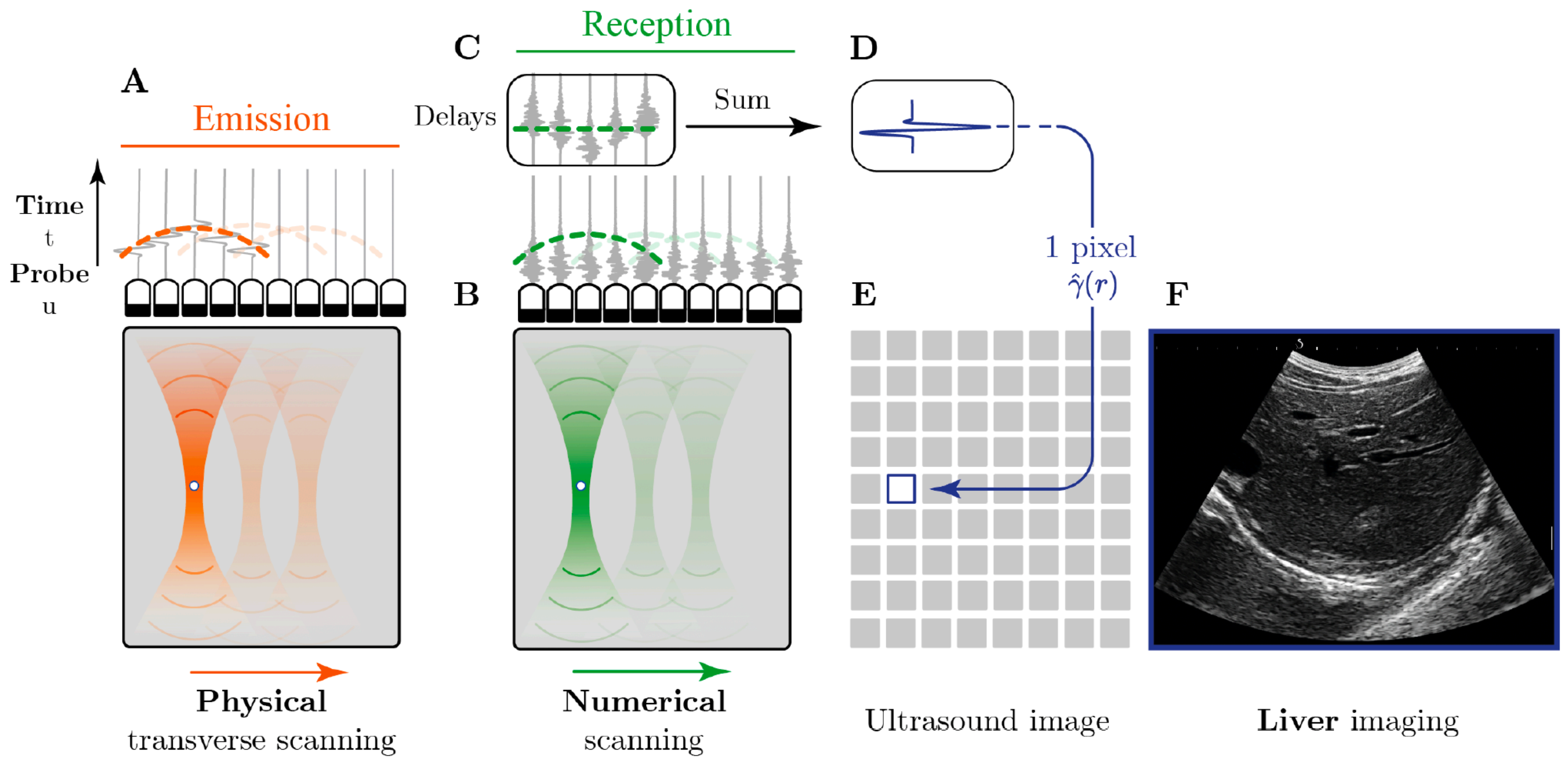


Image is computed using space time correspondance

In practice ?

Speed of sound in soft tissues varies ($\pm 10\,\%$) around a known value

Soft tissues are « weakly scattering »

Principle of ultrasound imaging

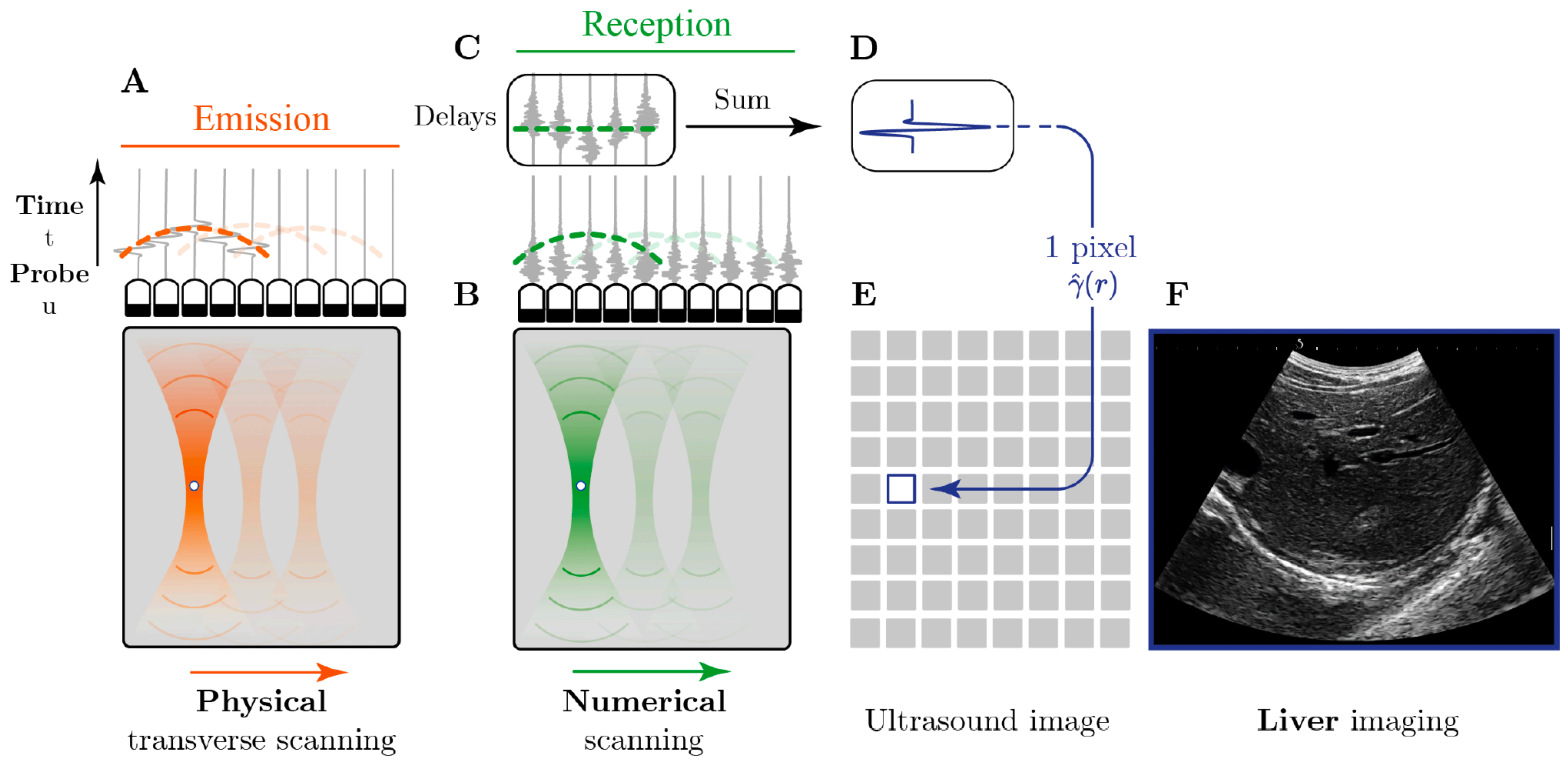


Image is computed using space time correspondance

In practice ?

Speed of sound in soft tissues varies ($\pm 10\,\%$) around a known value
Soft tissues are « weakly scattering »

Practical Hypotheses

Constant speed of sound
Single diffusion regime

Principle of ultrasound imaging

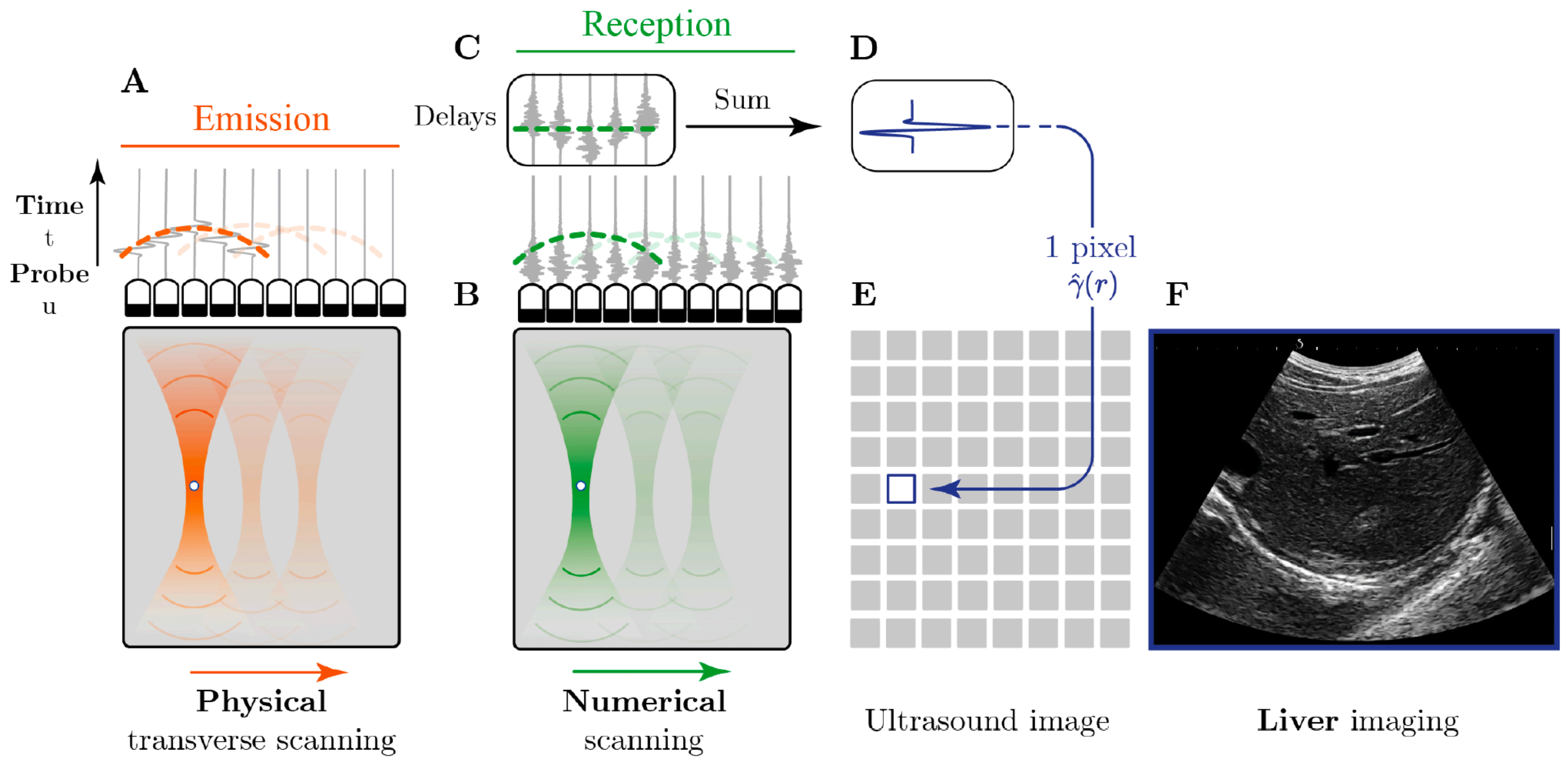


Image is computed using space time correspondance

In practice ?

Speed of sound in soft tissues varies ($\pm 10\,\%$) around a known value
Soft tissues are « weakly scattering »

Practical Hypotheses

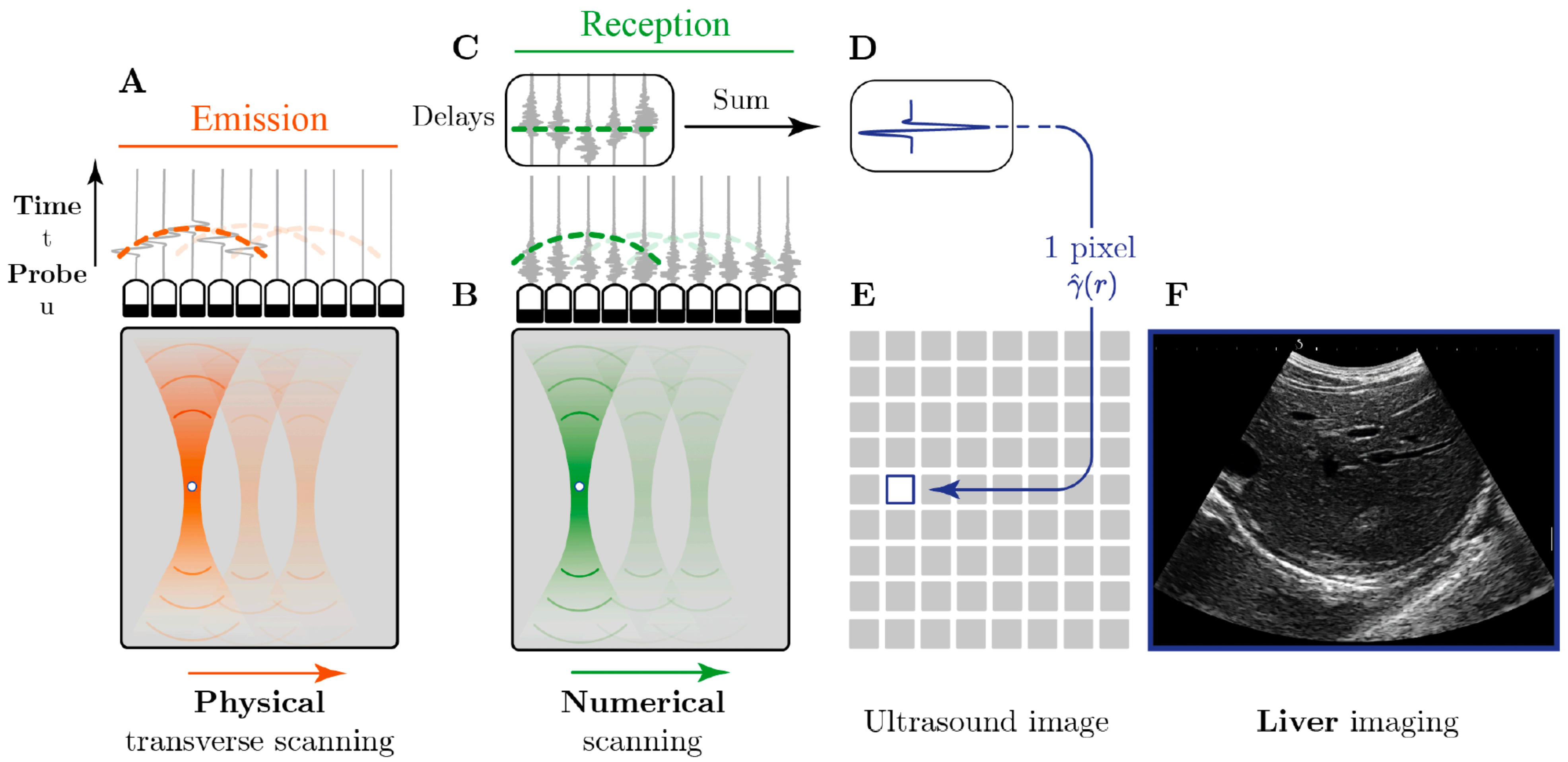
Constant speed of sound
Single diffusion regime

Improvements

Access to S.O.S → SNR, contrast, resolution, diagnosis..

Principle of ultrasound imaging

Image is computed using space time correspondance



In practice ?

Speed of sound in soft tissues varies ($\pm 10\,\%$) around a known value
Soft tissues are « weakly scattering »

Practical Hypotheses

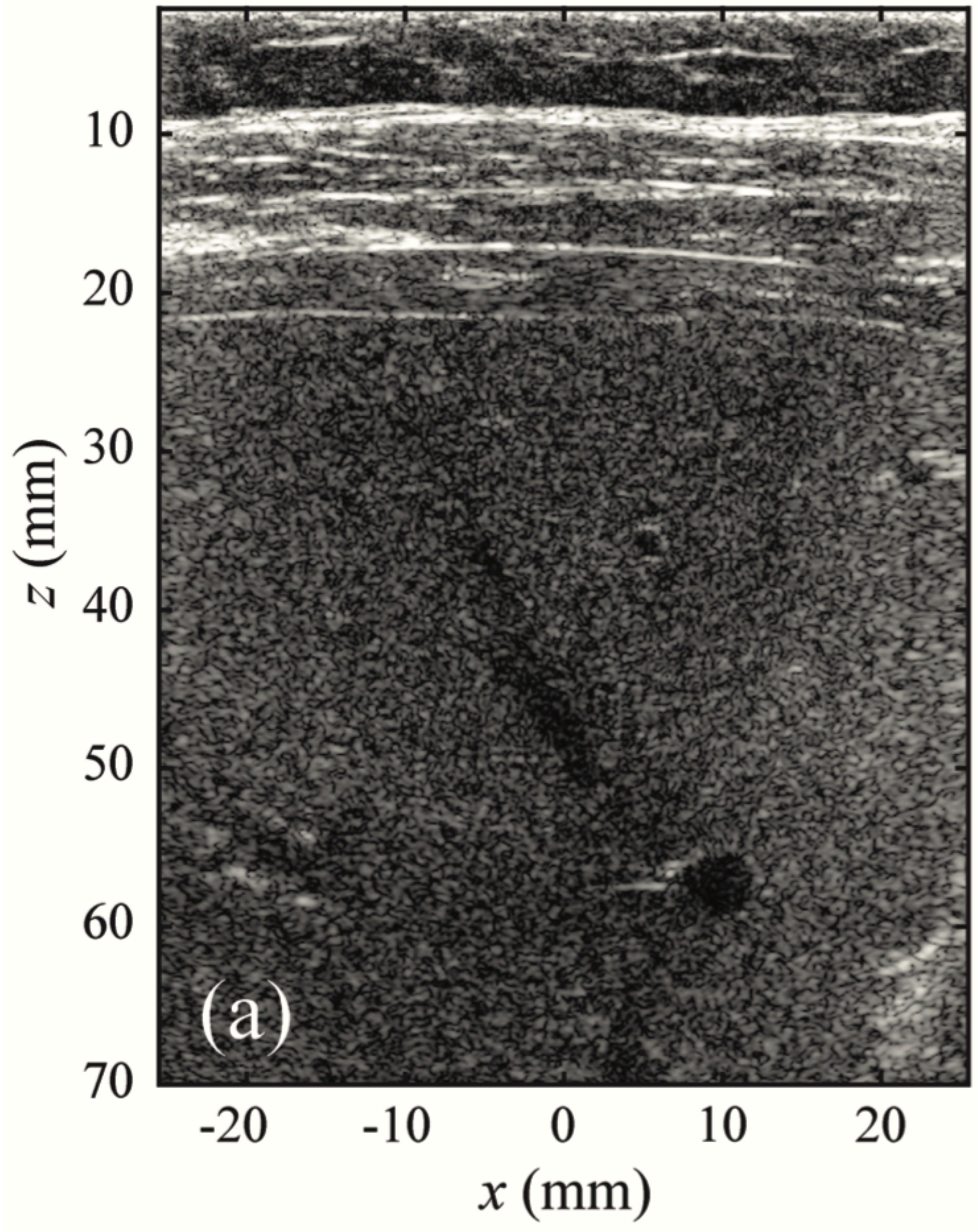
Constant speed of sound
Single diffusion regime

Improvements

Access to S.O.S → SNR, contrast, resolution, diagnosis..

Signal is the backscattered field → No easy access to travel times

Ultrasound images in soft tissues



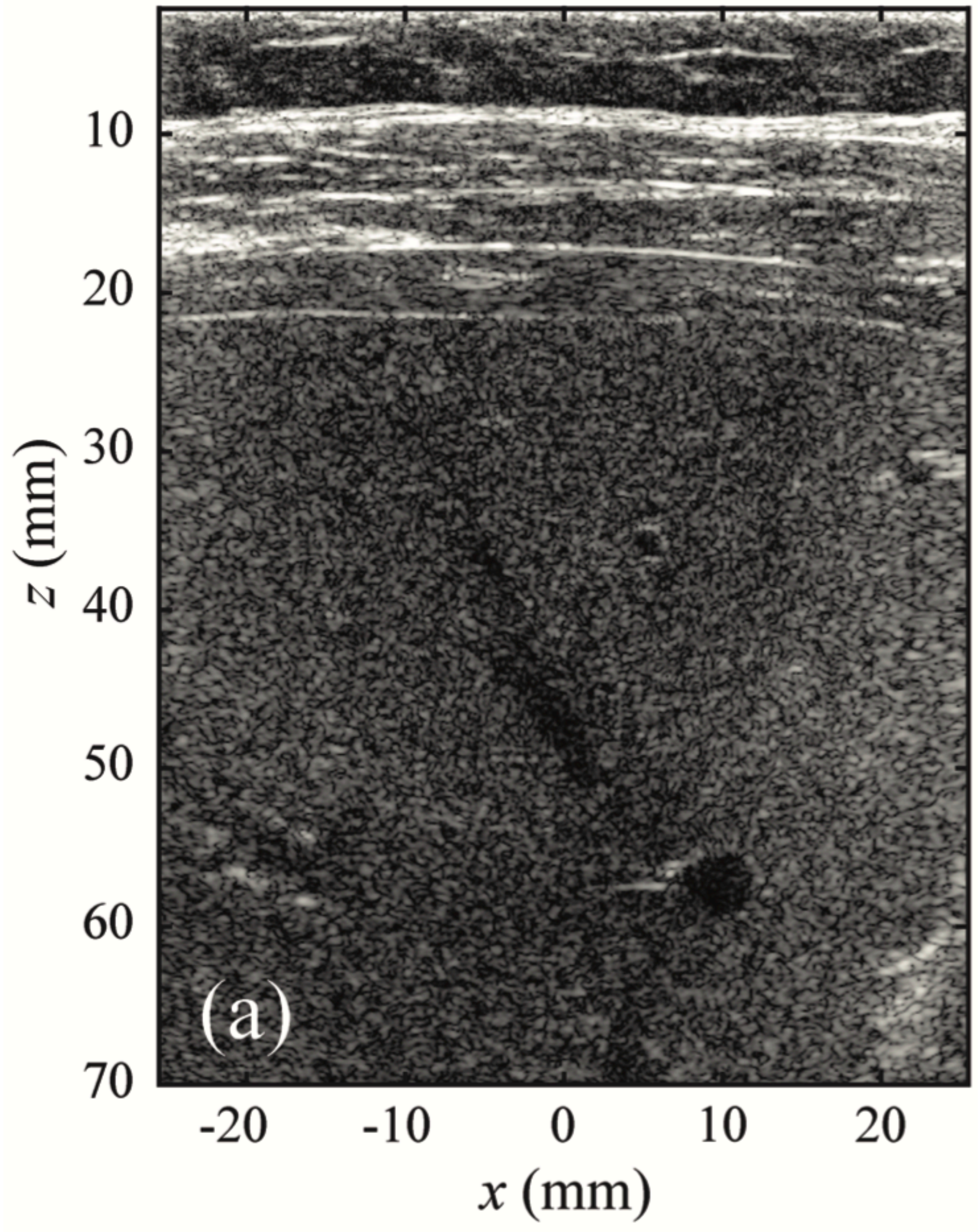
Ultrasound image of a liver
(W. Lambert)

Backscattered signal is generated by discontinuities of acoustic impedance $Z = \sqrt{\frac{\rho}{K}}$

Density

Compressibility

Ultrasound images in soft tissues



Ultrasound image of a liver
(W. Lambert)

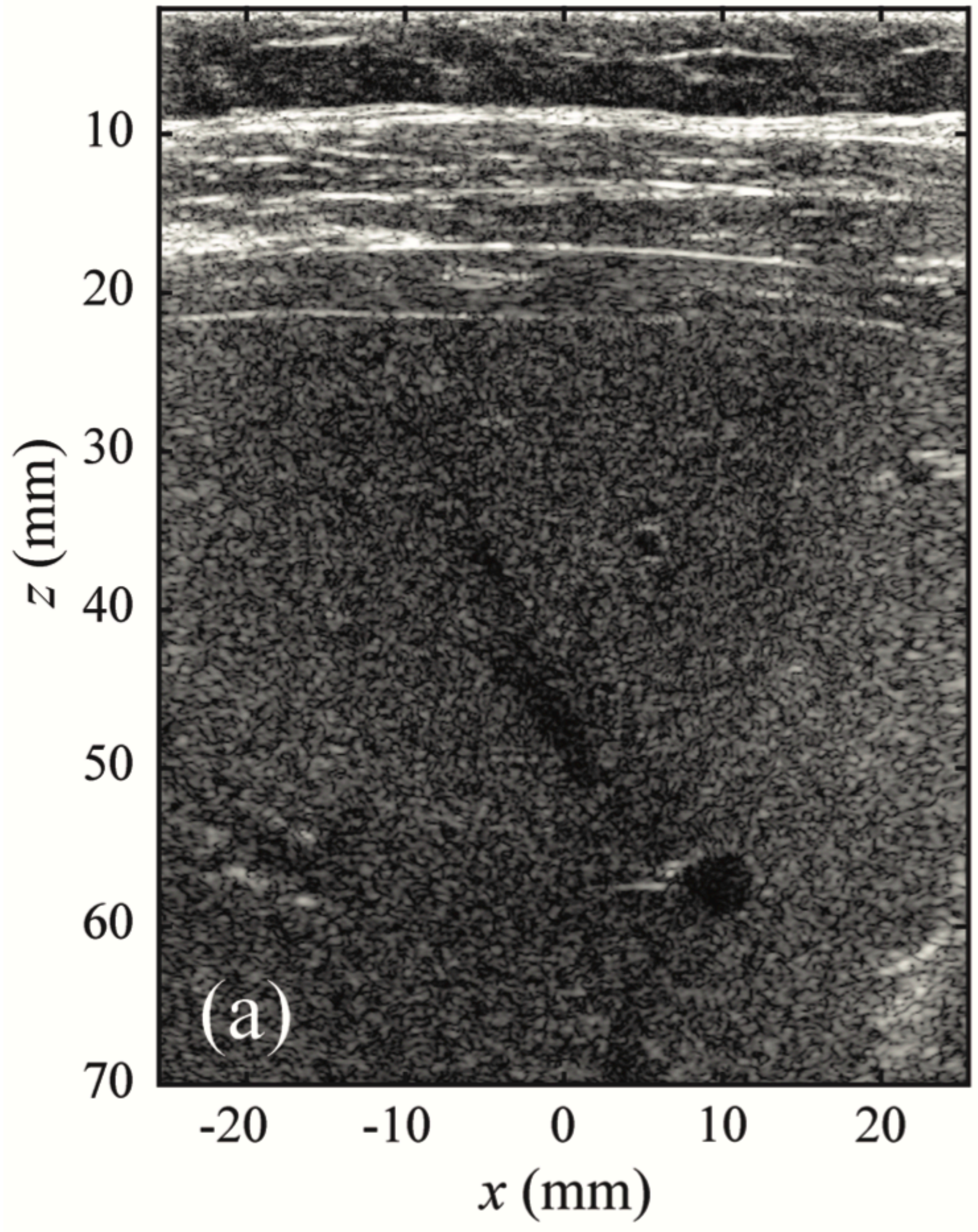
Backscattered signal is generated by discontinuities of acoustic impedance $Z = \sqrt{\frac{\rho}{K}}$

Density

Compressibility

Discontinuities appear at a microscopic scale (hundreds/wavelength), no isolated reflectors

Ultrasound images in soft tissues



Ultrasound image of a liver
(W. Lambert)

Backscattered signal is generated by discontinuities of acoustic impedance $Z = \sqrt{\frac{\rho}{K}}$

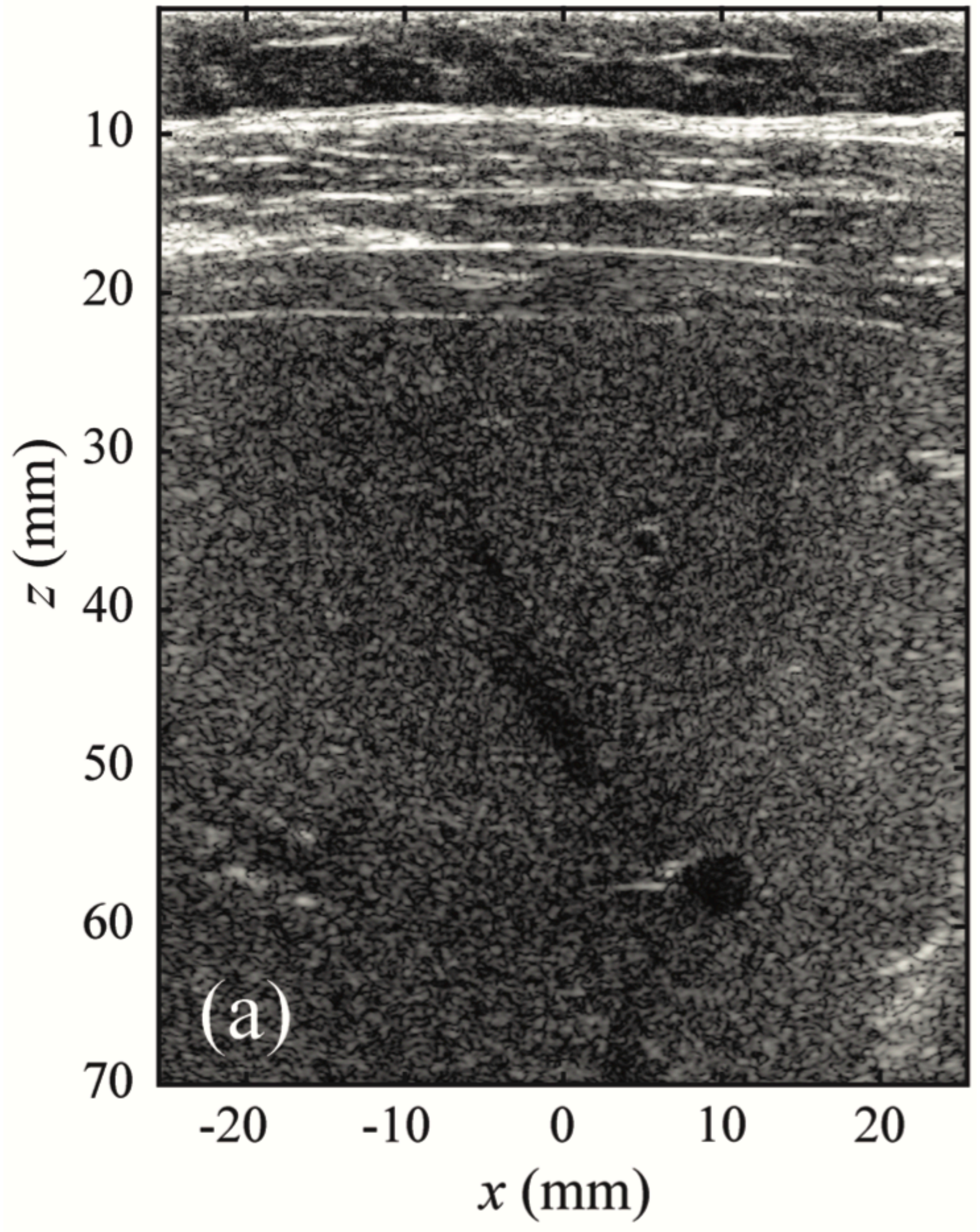
Density

Compressibility

Discontinuities appear at a microscopic scale (hundreds/wavelength), no isolated reflectors

Image is mainly speckle

Ultrasound images in soft tissues



Ultrasound image of a liver
(W. Lambert)

Backscattered signal is generated by discontinuities of acoustic impedance $Z = \sqrt{\frac{\rho}{K}}$

Density

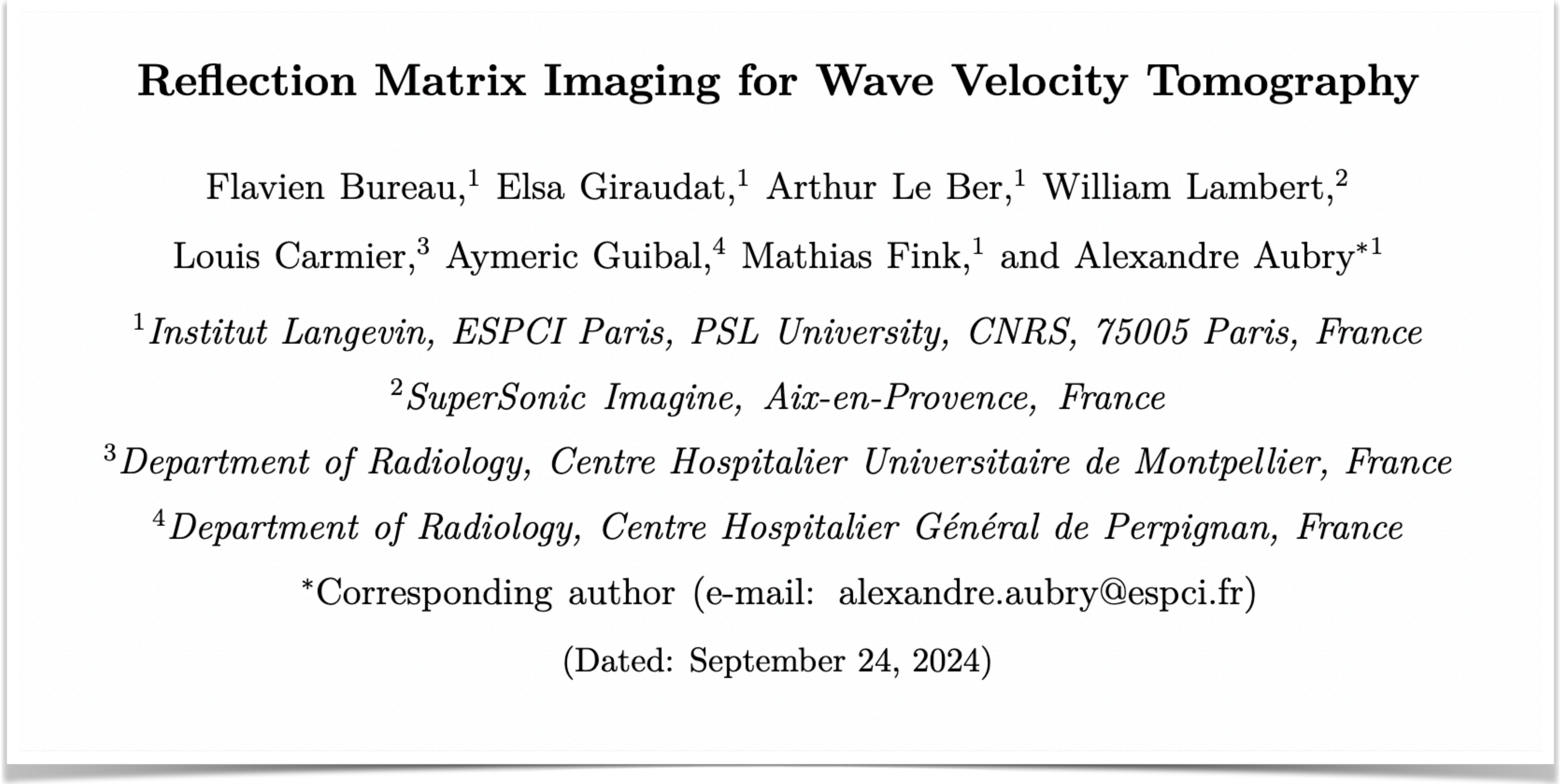
Compressibility

Discontinuities appear at a microscopic scale (hundreds/wavelength), no isolated reflectors

Image is mainly speckle

No access to the PSF

PSF probing in the speckle

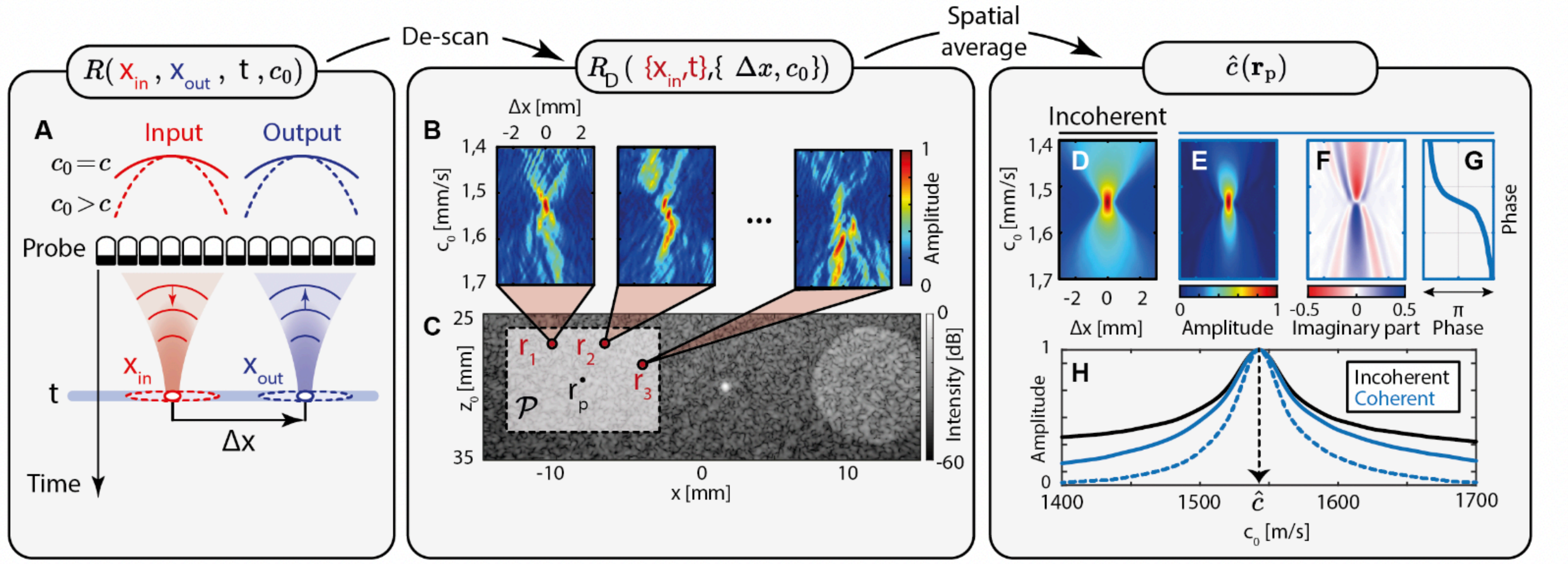


Reflection Matrix Imaging for Wave Velocity Tomography

Flavien Bureau,[1] Elsa Giraudat,[1] Arthur Le Ber,[1] William Lambert,[2]
Louis Carmier,[3] Aymeric Guibal,[4] Mathias Fink,[1] and Alexandre Aubry*[1]

[1] *Institut Langevin, ESPCI Paris, PSL University, CNRS, 75005 Paris, France*
[2] *SuperSonic Imagine, Aix-en-Provence, France*
[3] *Department of Radiology, Centre Hospitalier Universitaire de Montpellier, France*
[4] *Department of Radiology, Centre Hospitalier Général de Perpignan, France*
*Corresponding author (e-mail: alexandre.aubry@espci.fr)

(Dated: September 24, 2024)



Experimental work by A. Aubry and his team at Institut Langevin.

New method to reconstruct the propagation speed of ultrasounds in soft tissues inspired by adaptative optics

Provide a robust mathematical model for this inverse problem

Content

1. *MODEL / DIRECT PROBLEM*
2. *INVERSE PROBLEM*

Content

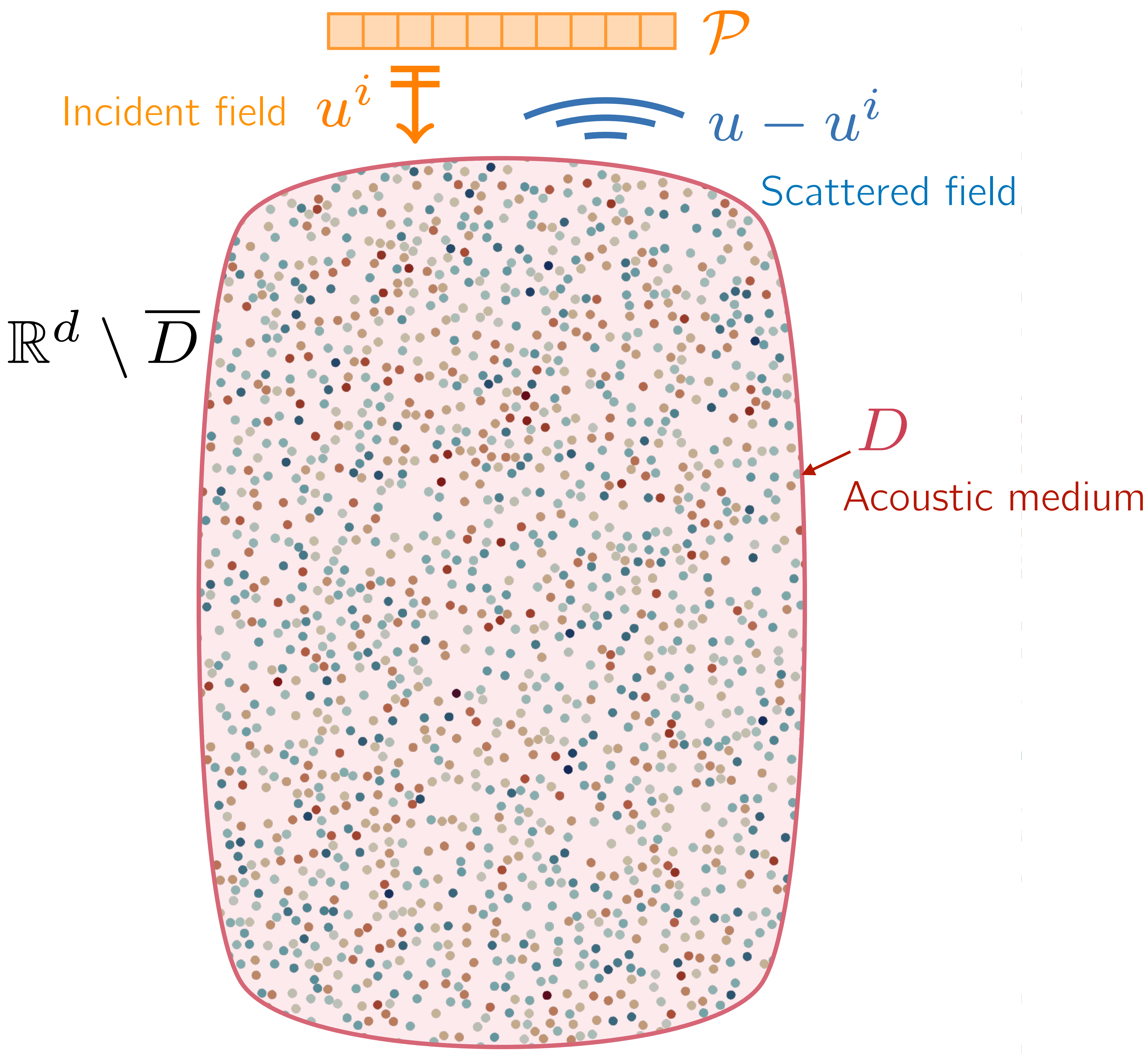
1. *MODEL / DIRECT PROBLEM*
2. *INVERSE PROBLEM*

What is is that we measure

What do we do with those measurements

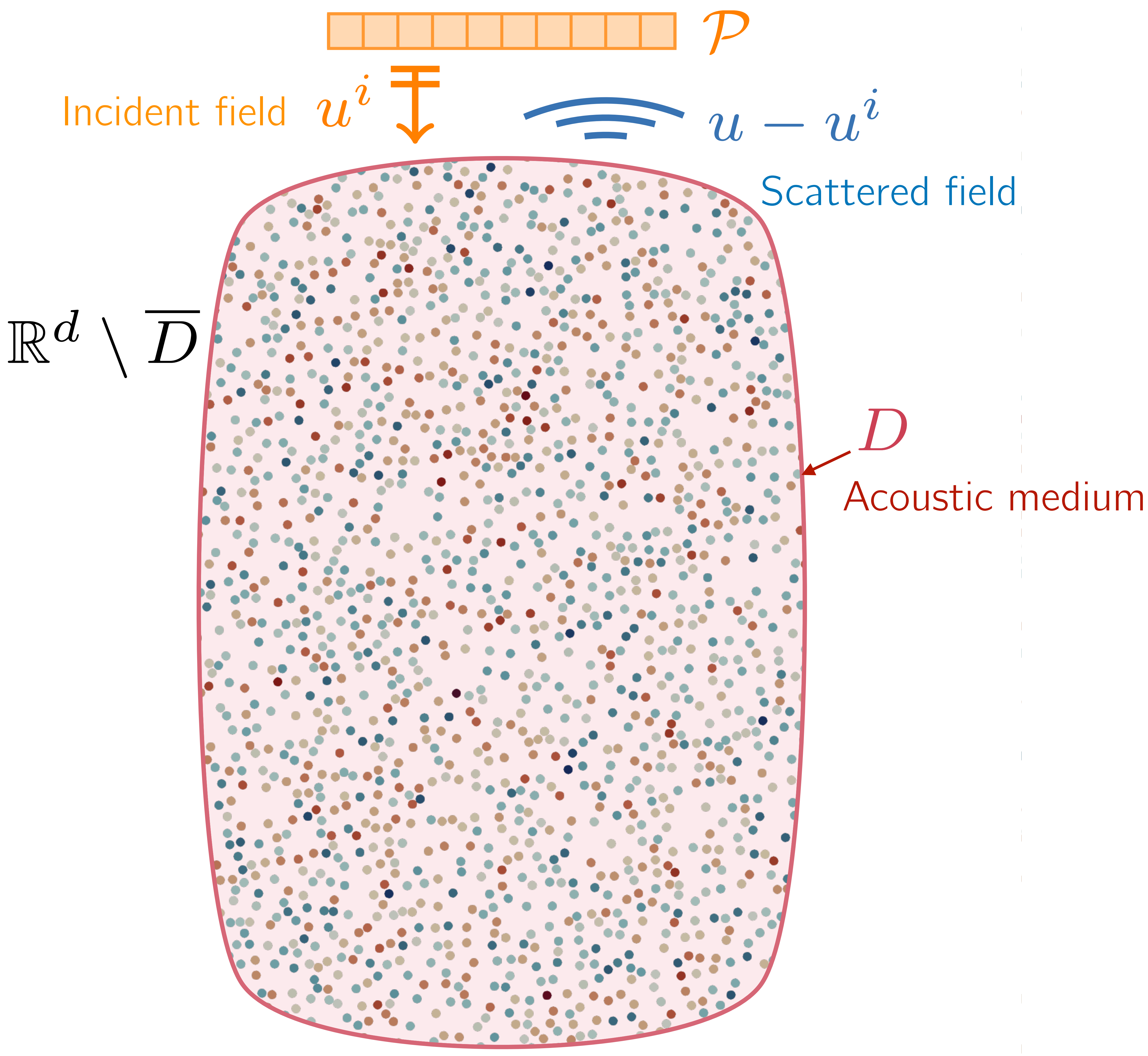
Content

The acoustic scattering problem



$$\begin{cases} \nabla \cdot (a \nabla u) + \omega^2 n u = 0 \\ \lim\limits_{|x| \to \infty} |x|^{\frac{d-1}{2}} \left(\partial_{|x|} \left(u - u^i \right) - i k_0 \left(u - u^i \right) \right) = 0. \end{cases}$$

The acoustic scattering problem



$$\begin{cases} \nabla \cdot (a \nabla u) + \omega^2 n u = 0 \\ \lim\limits_{|x| \to \infty} |x|^{\frac{d-1}{2}} \left(\partial_{|x|} \left(u - u^i \right) - i k_0 \left(u - u^i \right) \right) = 0. \end{cases}$$

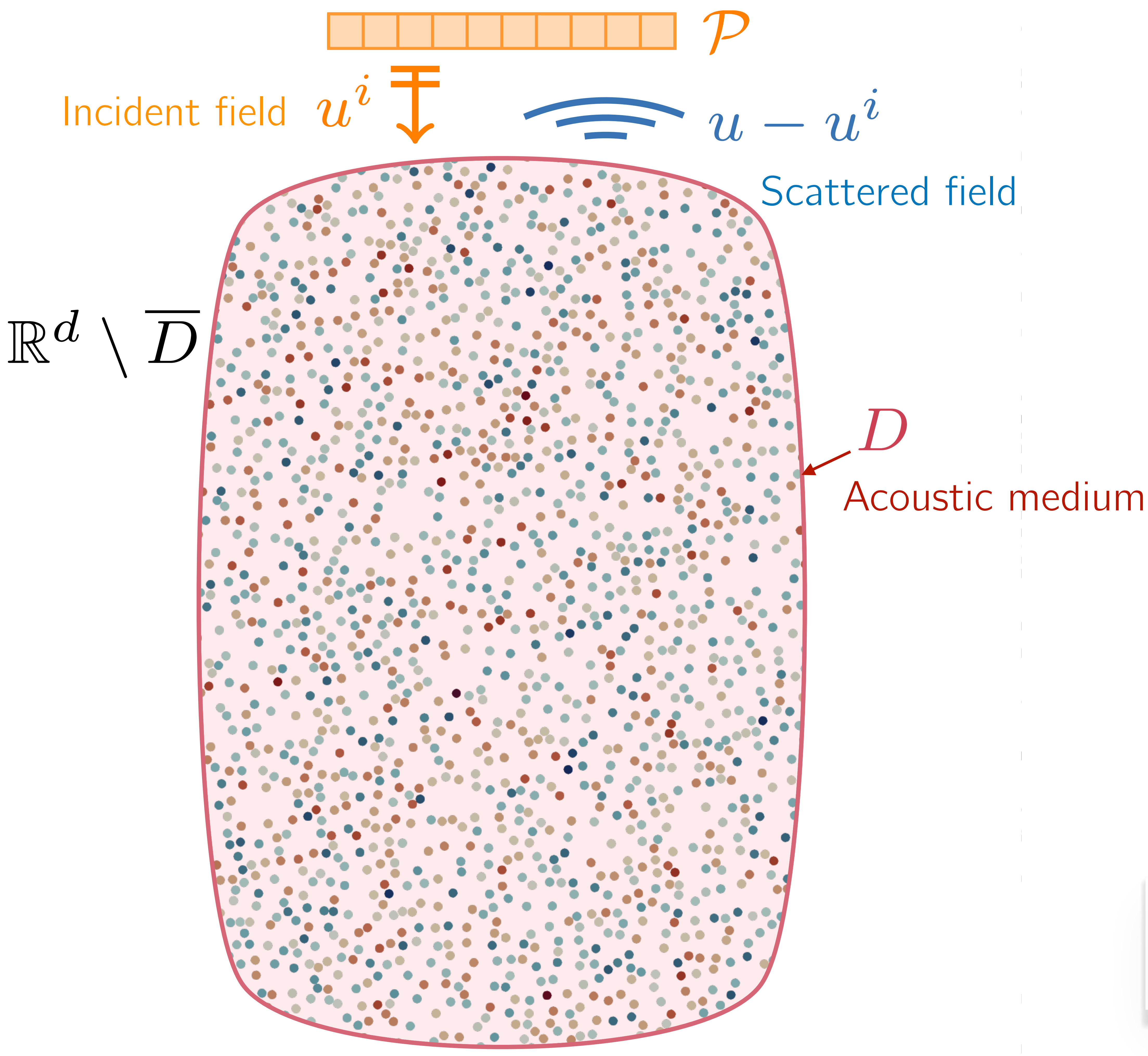
Physical parameters:

$$a(x) := a_0 \mathbb{1}_{\mathbb{R}^d \setminus \overline{D}} + a_D(x) \mathbb{1}_D,$$

$$n(x) := n_0 \mathbb{1}_{\mathbb{R}^d \setminus \overline{D}} + n_D(x) \mathbb{1}_D.$$

$$k_0 := \omega \sqrt{n_0 a_0^{-1}}$$

The acoustic scattering problem



$$\begin{cases} \nabla \cdot (a \nabla u) + \omega^2 n u = 0 \\ \lim\limits_{|x| \to \infty} |x|^{\frac{d-1}{2}} \left(\partial_{|x|} \left(u - u^i\right) - i k_0 \left(u - u^i\right)\right) = 0. \end{cases}$$

Physical parameters:

$$a(x) := a_0 \mathbb{1}_{\mathbb{R}^d \setminus \overline{D}} + a_D(x) \mathbb{1}_D,$$

$$n(x) := n_0 \mathbb{1}_{\mathbb{R}^d \setminus \overline{D}} + n_D(x) \mathbb{1}_D.$$

$$k_0 := \omega \sqrt{n_0 a_0^{-1}}$$

What is a good model for $a(x)$ and $n(x)$?

Model for soft tissues

Soft tissues are incredibly complex

Realistic model for $a(x)$ and $n(x)$ is a lost cause

Model for soft tissues

Soft tissues are incredibly complex

Realistic model for $a(x)$ and $n(x)$ is a lost cause

Tissue-mimicking phantoms are much simpler :

LOTS of small scatterers densely embedded in an homogeneous matrix

Model for soft tissues

Soft tissues are incredibly complex

Realistic model for $a(x)$ and $n(x)$ is a lost cause

Tissue-mimicking phantoms are much simpler :

LOTS of small scatterers densely embedded in an homogeneous matrix

Variation in speed of sound obtained by locally changing the repartition of scatterers

Mathematical framework requirements

Random
Stable w/ respect to size and number of scatterers
Can infer macroscopic quantities from the micro-structure

Model for soft tissues

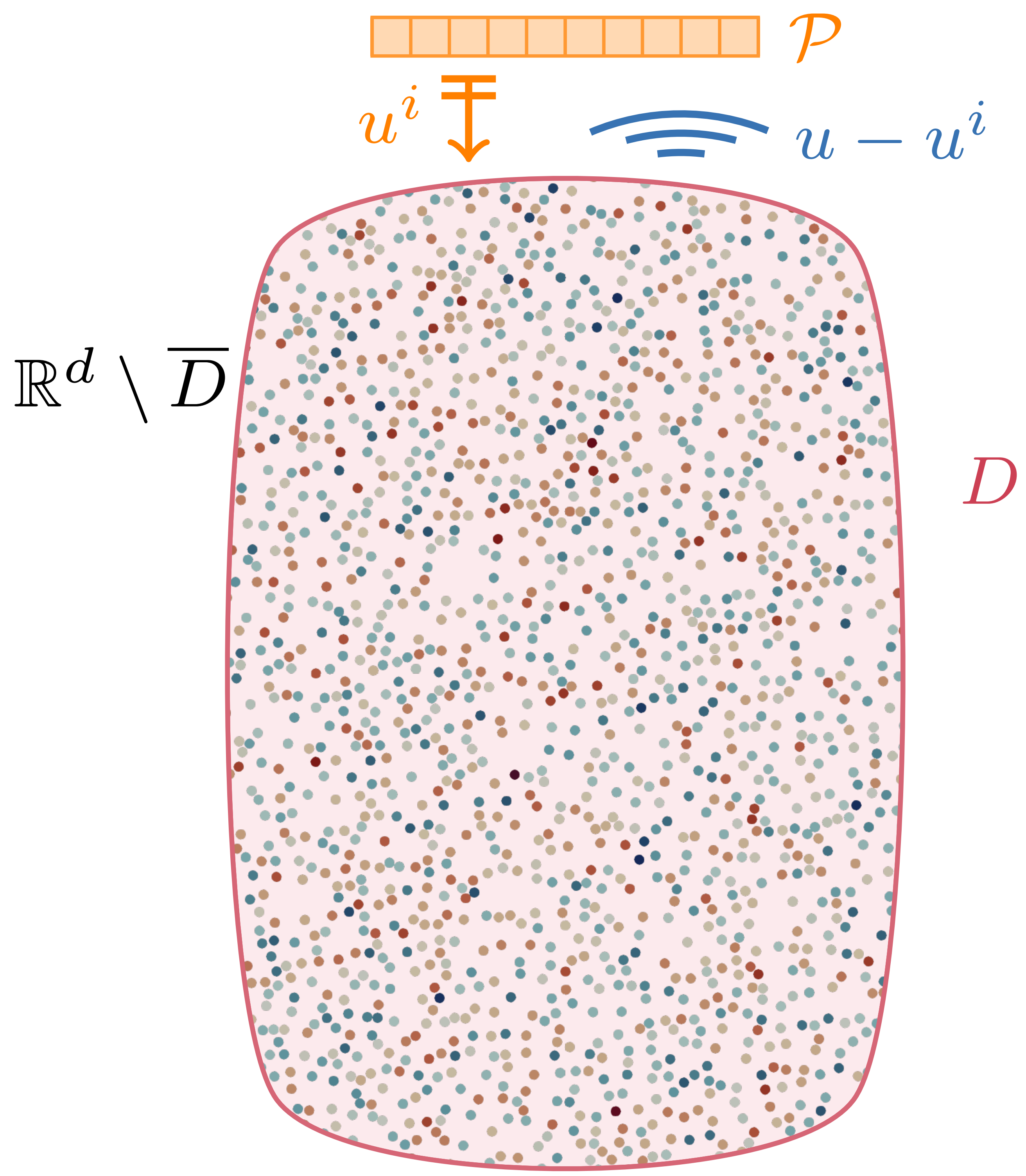
Soft tissues are incredibly complex

Realistic model for $a(x)$ and $n(x)$ is a lost cause

Tissue-mimicking phantoms are much simpler :

LOTS of small scatterers densely embedded in an homogeneous matrix

Variation in speed of sound obtained by locally changing the repartition of scatterers



Mathematical framework requirements

Random
Stable w/ respect to size and number of scatterers
Can infer macroscopic quantities from the micro-structure

Stochastic Homogenization

Mathematical description of the micro-structured medium

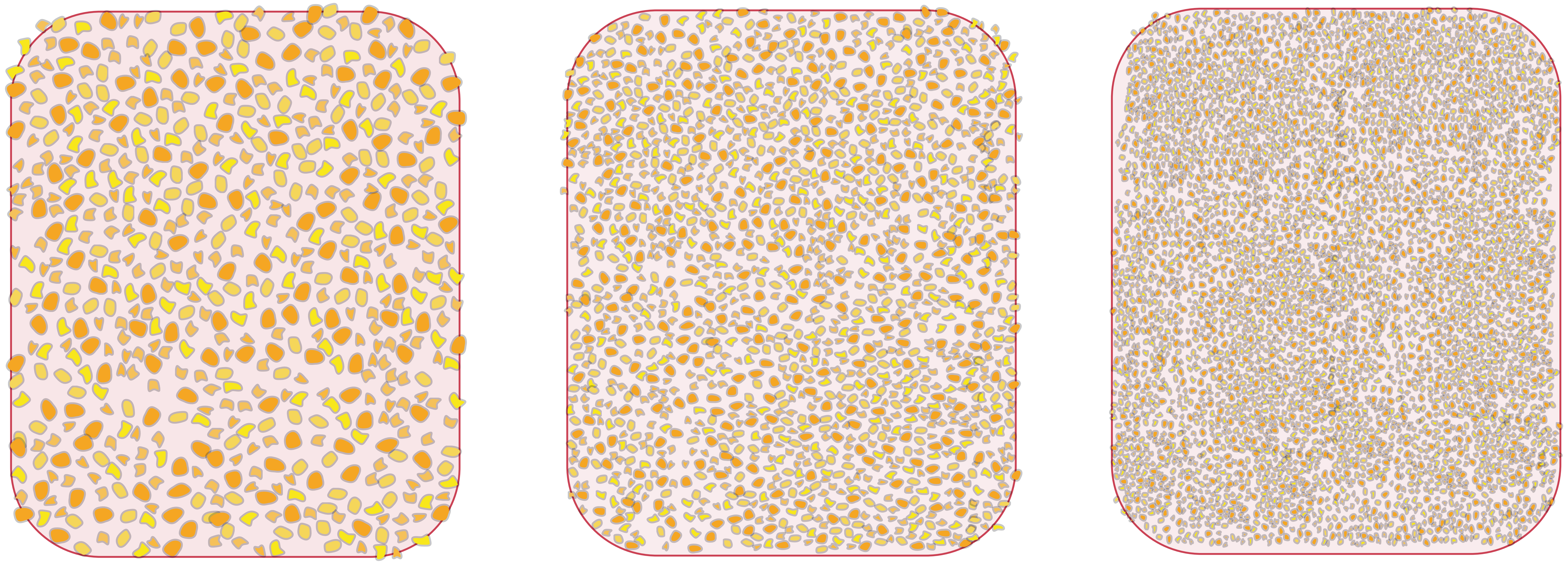
$\mathcal{Q}_i$ Domains of measure 1

x_i Random centers distributed in $\mathbb{R}^d$

$S = \cup_i \{x_i + \mathcal{Q}\}$ Set of size 1 scatterers

$$\begin{cases} a_1 := a_M \mathbb{1}_{\mathbb{R}^d \setminus \overline{S}} + a_S \mathbb{1}_S, \\ n_1 := n_M \mathbb{1}_{\mathbb{R}^d \setminus \overline{S}} + n_S \mathbb{1}_S. \end{cases}$$

$$\forall x \in D \begin{cases} a_D^\varepsilon(x) = a_1 \left(\frac{x}{\varepsilon}\right), \\ n_D^\varepsilon(x) := n_1 \left(\frac{x}{\varepsilon}\right). \end{cases}$$



Example with $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$

Mathematical description of the micro-structured medium

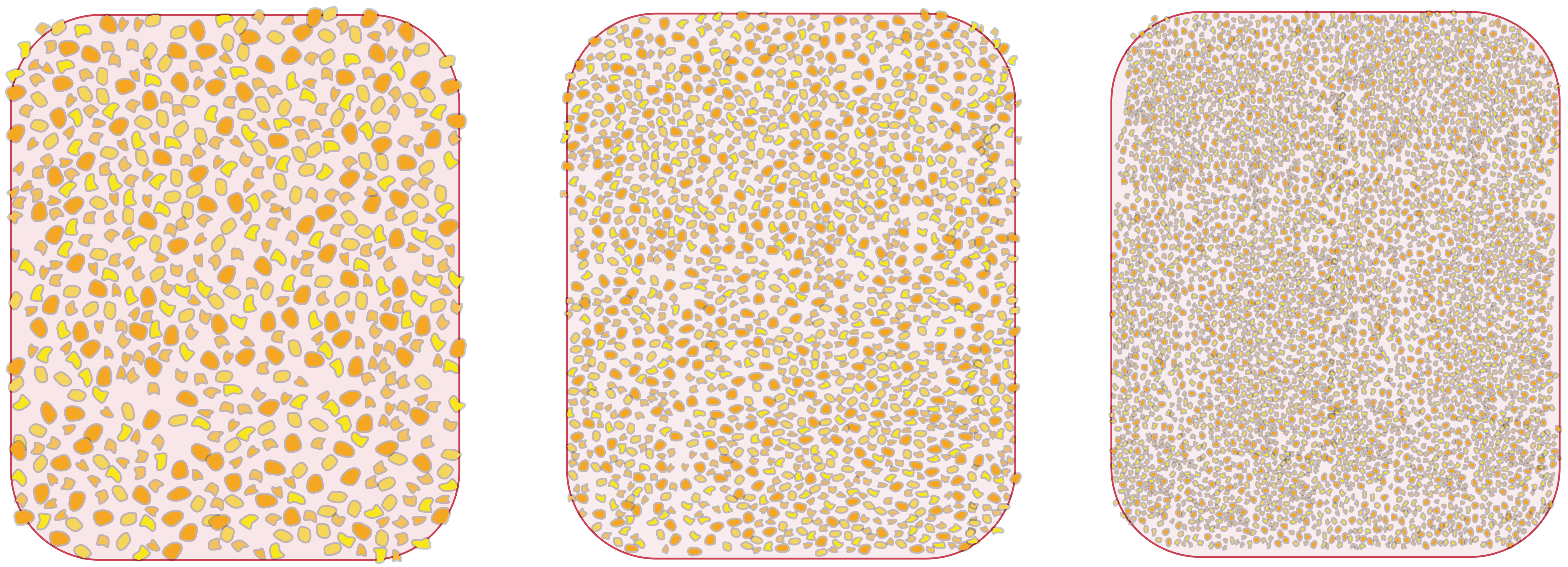
$\mathcal{Q}_i$ Domains of measure 1

x_i Random centers distributed in $\mathbb{R}^d$

$S = \cup_i \{x_i + \mathcal{Q}\}$ Set of size 1 scatterers

$$\begin{cases} a_1 := a_M \mathbb{1}_{\mathbb{R}^d \setminus \overline{S}} + a_S \mathbb{1}_S, \\ n_1 := n_M \mathbb{1}_{\mathbb{R}^d \setminus \overline{S}} + n_S \mathbb{1}_S. \end{cases}$$

$$\forall x \in D \begin{cases} a_D^\varepsilon(x) = a_1\left(\frac{x}{\varepsilon}\right), \\ n_D^\varepsilon(x) := n_1\left(\frac{x}{\varepsilon}\right). \end{cases}$$



Example with $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$

Simplification $a \equiv 1$

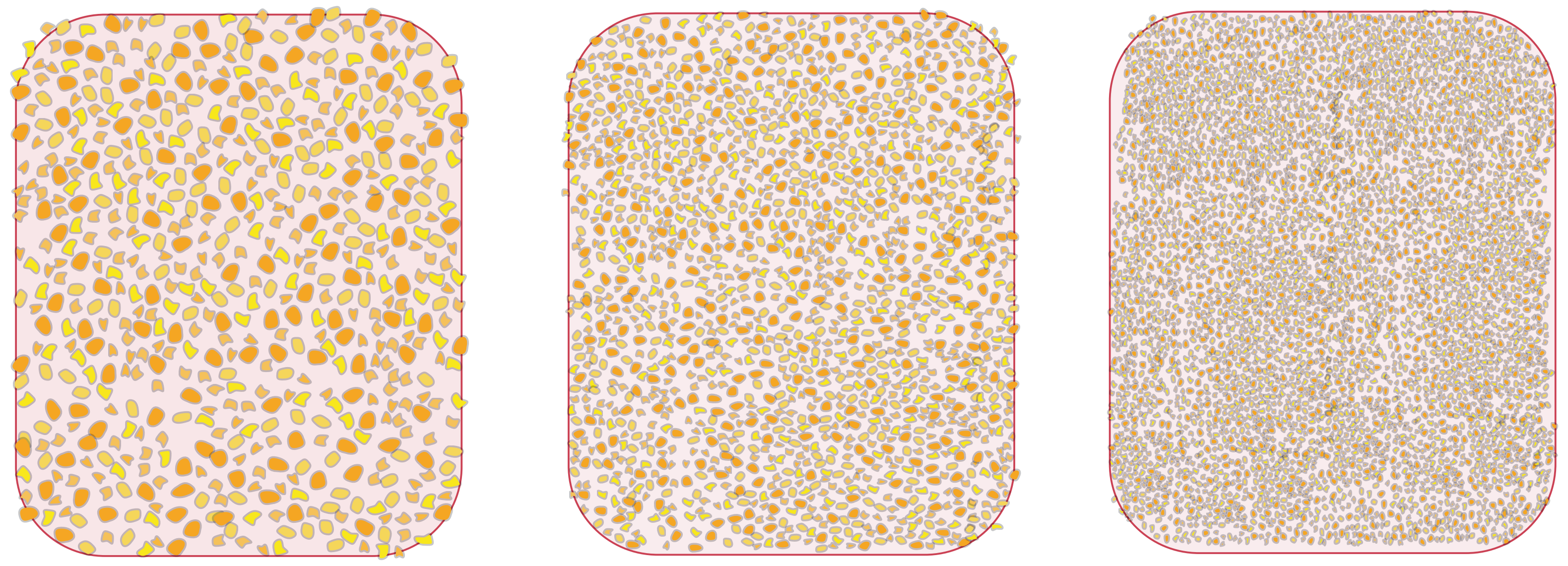
Mathematical description of the micro-structured medium

$\mathcal{Q}_i$ Domains of measure 1
x_i Random centers distributed in $\mathbb{R}^d$

$S = \cup_i \{x_i + \mathcal{Q}\}$ Set of size 1 scatterers

$$\begin{cases} a_1 := a_M \mathbb{1}_{\mathbb{R}^d \setminus \overline{S}} + a_S \mathbb{1}_S, \\ n_1 := n_M \mathbb{1}_{\mathbb{R}^d \setminus \overline{S}} + n_S \mathbb{1}_S. \end{cases}$$

$$\forall x \in D \begin{cases} a_D^\varepsilon(x) = a_1\left(\frac{x}{\varepsilon}\right), \\ n_D^\varepsilon(x) := n_1\left(\frac{x}{\varepsilon}\right). \end{cases}$$



Example with $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$

For the case $a \not\equiv 1$

see L. Giovangigli's talk

Simplification $a \equiv 1$

The coherent field

Governing equation total field u

$$\begin{cases} \Delta u + \omega^2 (n_0 + (n_\varepsilon - n_0)\mathbb{1}_D) u = 0 \\ \lim\limits_{|x|\to\infty} |x|^{\frac{d-1}{2}} \left(\partial_{|x|}\left(u - u^i\right) - ik_0\left(u - u^i\right)\right) = 0. \end{cases}$$

The coherent field

Governing equation total field u

$$\begin{cases} \Delta u + \omega^2(n_0 + (n_\varepsilon - n_0)\mathbb{1}_D)u = 0 \\ \lim\limits_{|x|\to\infty} |x|^{\frac{d-1}{2}} \left(\partial_{|x|}\left(u - u^i\right) - ik_0\left(u - u^i\right)\right) = 0. \end{cases}$$

Governing equation coherent field u_0

$$\begin{cases} \Delta u_0 + \omega^2(n_0 + (n^\star - n_0)\mathbb{1}_D)u_0 = 0 \\ \lim\limits_{|x|\to\infty} |x|^{\frac{d-1}{2}} \left(\partial_{|x|}\left(u_0 - u^i\right) - ik_0\left(u_0 - u^i\right)\right) = 0. \end{cases}$$

Effective index in D

$$n^\star \equiv \mathbb{E}[n_\varepsilon]$$

The coherent field

Governing equation total field u

$$\begin{cases} \Delta u + \omega^2 (n_0 + (n_\varepsilon - n_0) \mathbb{1}_D) u = 0 \\ \lim\limits_{|x| \to \infty} |x|^{\frac{d-1}{2}} \left(\partial_{|x|} \left(u - u^i \right) - i k_0 \left(u - u^i \right) \right) = 0. \end{cases}$$

Governing equation coherent field u_0

$$\begin{cases} \Delta u_0 + \omega^2 (n_0 + (n^\star - n_0) \mathbb{1}_D) u_0 = 0 \\ \lim\limits_{|x| \to \infty} |x|^{\frac{d-1}{2}} \left(\partial_{|x|} \left(u_0 - u^i \right) - i k_0 \left(u_0 - u^i \right) \right) = 0. \end{cases}$$

Effective index in D

$$n^\star \equiv \mathbb{E}[n_\varepsilon]$$

Speed of sound outside of D

$$k_0 := \omega \sqrt{n_0} := \frac{\omega}{c_0}$$

$$k^\star := \omega \sqrt{n^\star} := \frac{\omega}{c^\star}$$

Effective speed of sound in D

The coherent field

Governing equation total field u

$$\begin{cases} \Delta u + \omega^2 (n_0 + (n_\varepsilon - n_0) \mathbb{1}_D) u = 0 \\ \lim\limits_{|x| \to \infty} |x|^{\frac{d-1}{2}} \left(\partial_{|x|} \left(u - u^i \right) - i k_0 \left(u - u^i \right) \right) = 0. \end{cases}$$

Governing equation coherent field u_0

$$\begin{cases} \Delta u_0 + \omega^2 (n_0 + (n^\star - n_0) \mathbb{1}_D) u_0 = 0 \\ \lim\limits_{|x| \to \infty} |x|^{\frac{d-1}{2}} \left(\partial_{|x|} \left(u_0 - u^i \right) - i k_0 \left(u_0 - u^i \right) \right) = 0. \end{cases}$$

Effective index in D

$$n^\star \equiv \mathbb{E}[n_\varepsilon]$$

Homogenized Green function

$$\begin{cases} \Delta G^\star(\cdot, y) + \omega^2 (n_0 + (n^\star - n_0) \mathbb{1}_D) G^\star(\cdot, y) = \delta_y \\ \lim\limits_{|x| \to \infty} |x|^{\frac{d-1}{2}} \left(\partial_{|x|} G^\star(x, y) - i k_0 G^\star(x, y) \right) = 0. \end{cases}$$

Speed of sound outside of D

$$k_0 := \omega \sqrt{n_0} := \frac{\omega}{c_0}$$

$$k^\star := \omega \sqrt{n^\star} := \frac{\omega}{c^\star}$$

Effective speed of sound in D

The scattered field

$$\mathcal{U}^s := \omega^2 \int_D \left(n_\varepsilon(y) - n^\star\right) u_0(y) G^\star(\cdot, y)\mathrm{d}y.$$

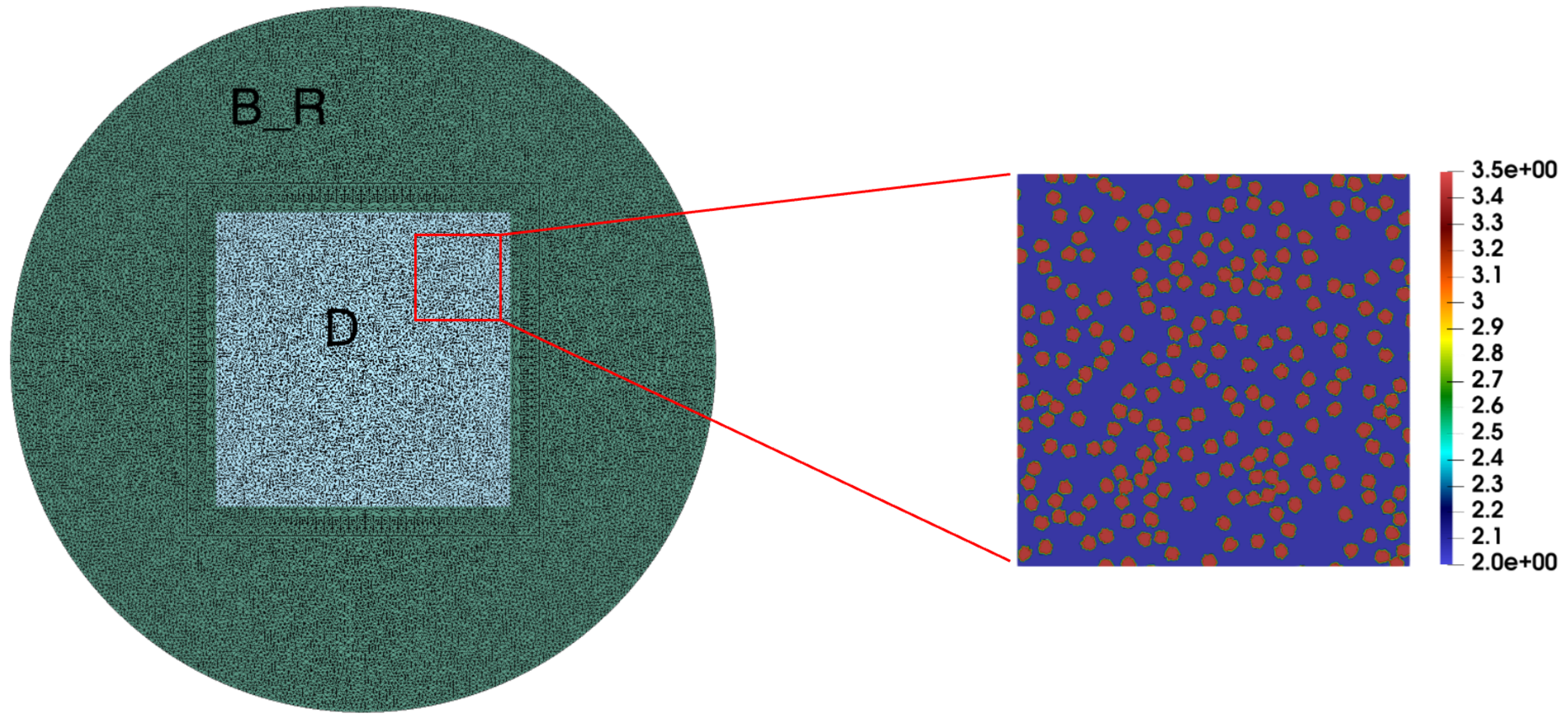
Quantitative Homogenization Theorem (Garnier, Giovangigli, Goepfert, M.)

$$\|u(x) - u^i(x) - \mathcal{U}^s(x)\|_{L^2(\Omega)} \leq C_\omega \varepsilon^{\frac{d}{p'}} \qquad \forall x \in \mathcal{P}$$

C_ω indep. of ε and x. $p' = \dfrac{2+\beta}{1+\beta}$ w/ $\sup_{x\in D} \int_D |G^\star(x,y)|^{2+\beta} \mathrm{d}y < \infty$

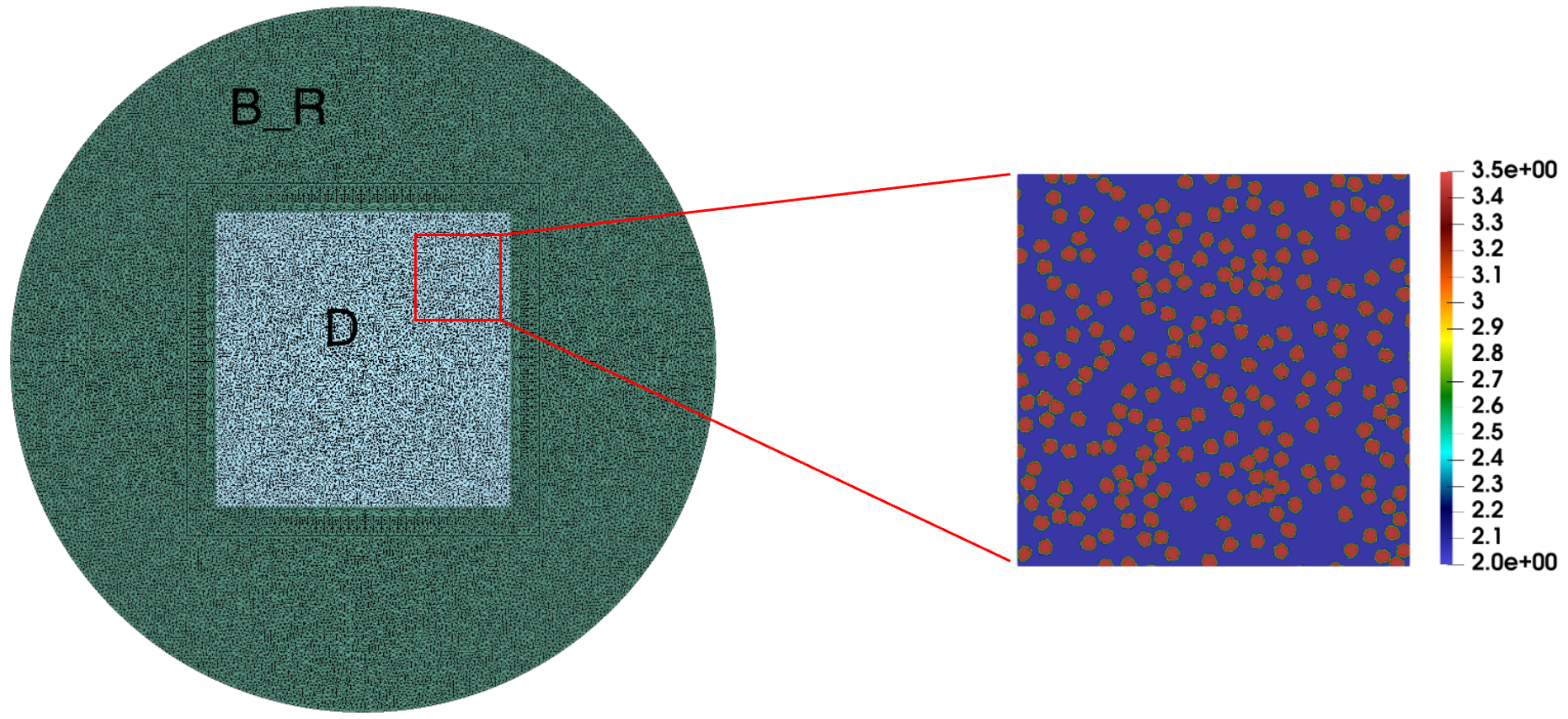
Illustrations

Example of domain

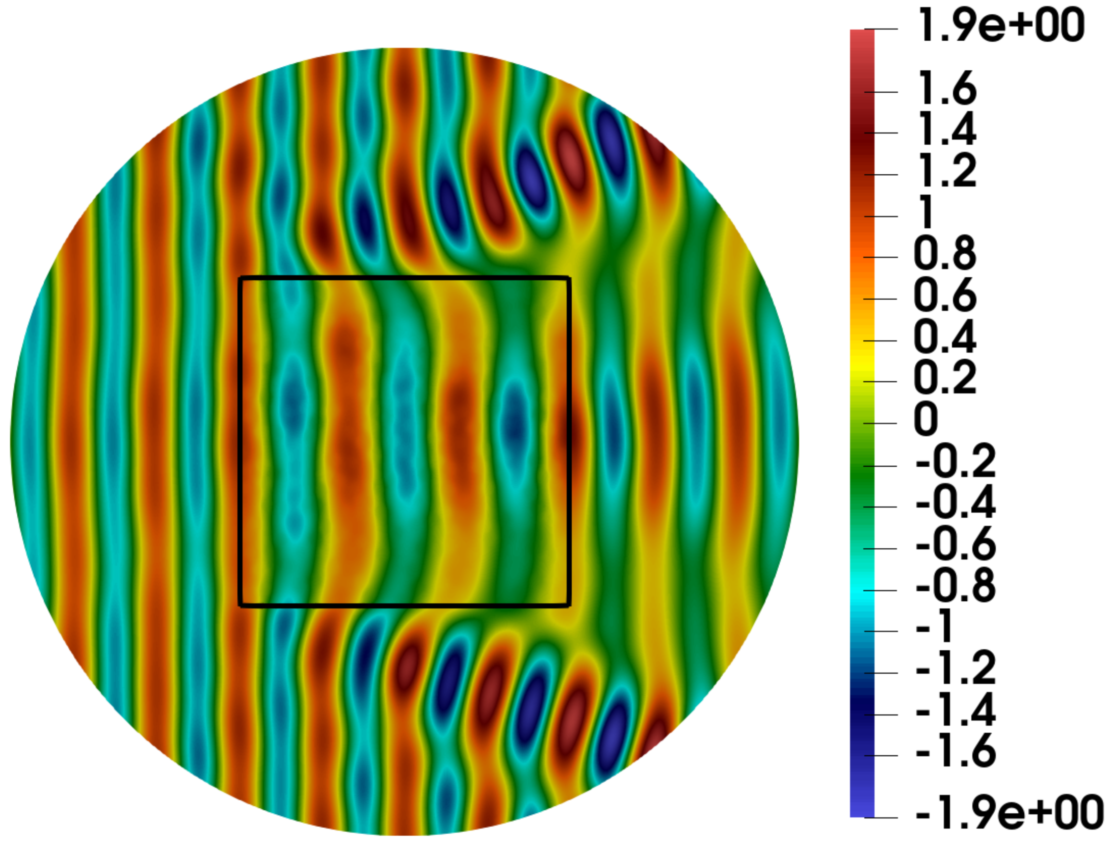


Illustrations

Example of domain

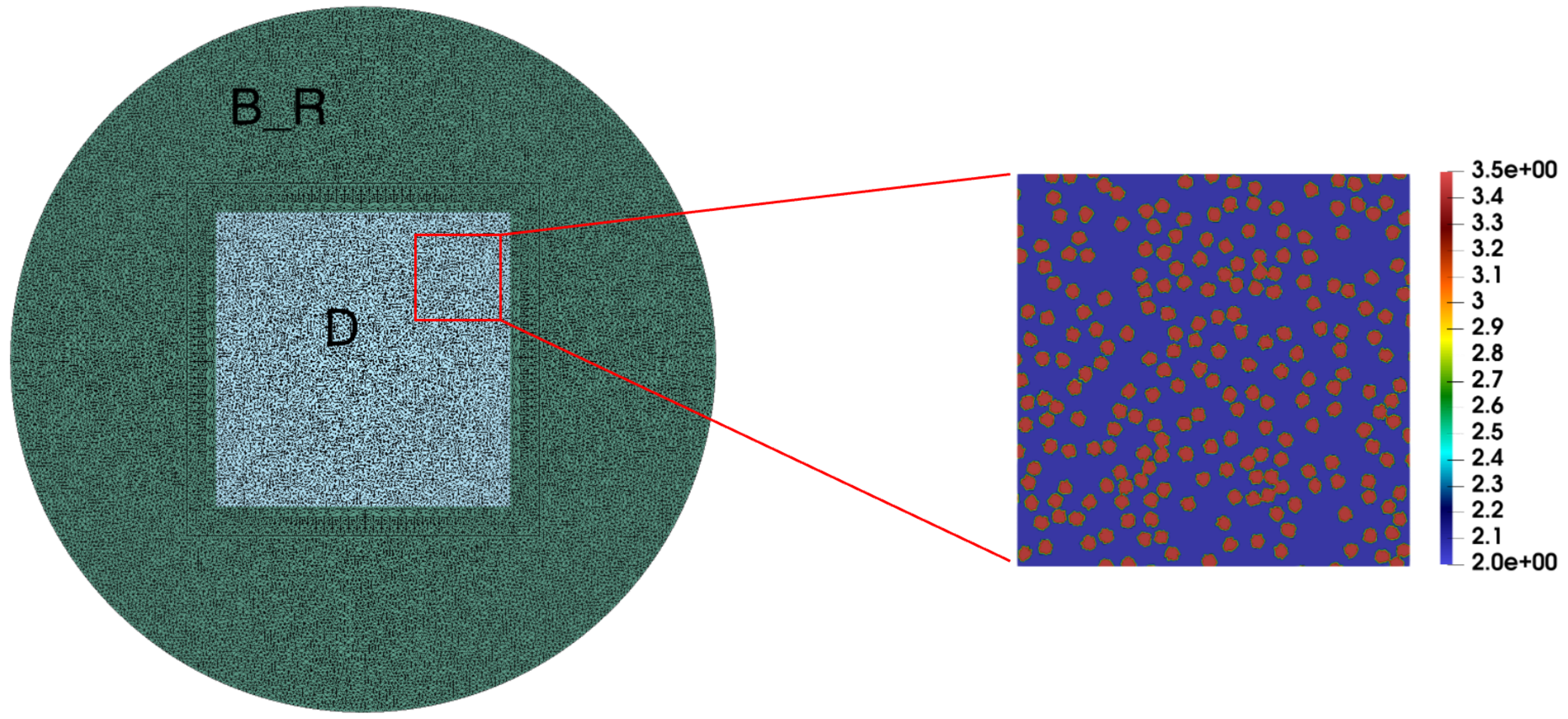


Total field u^ε

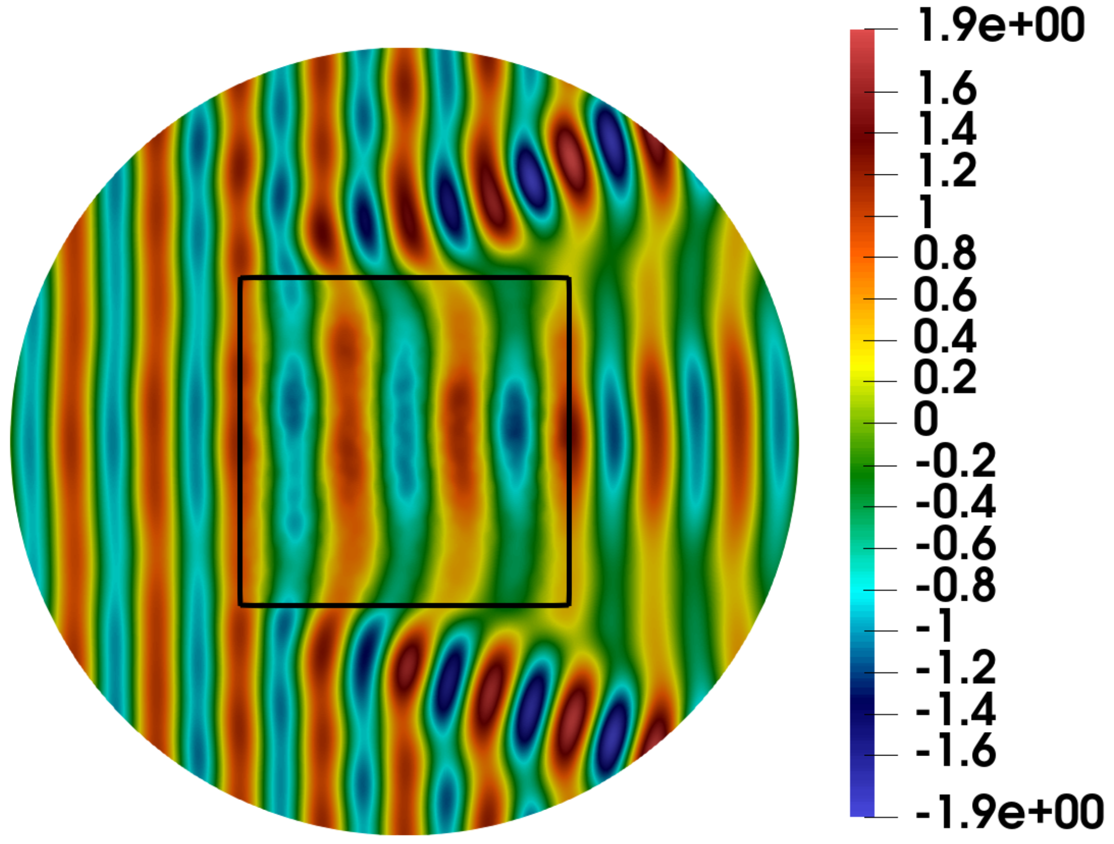


Illustrations

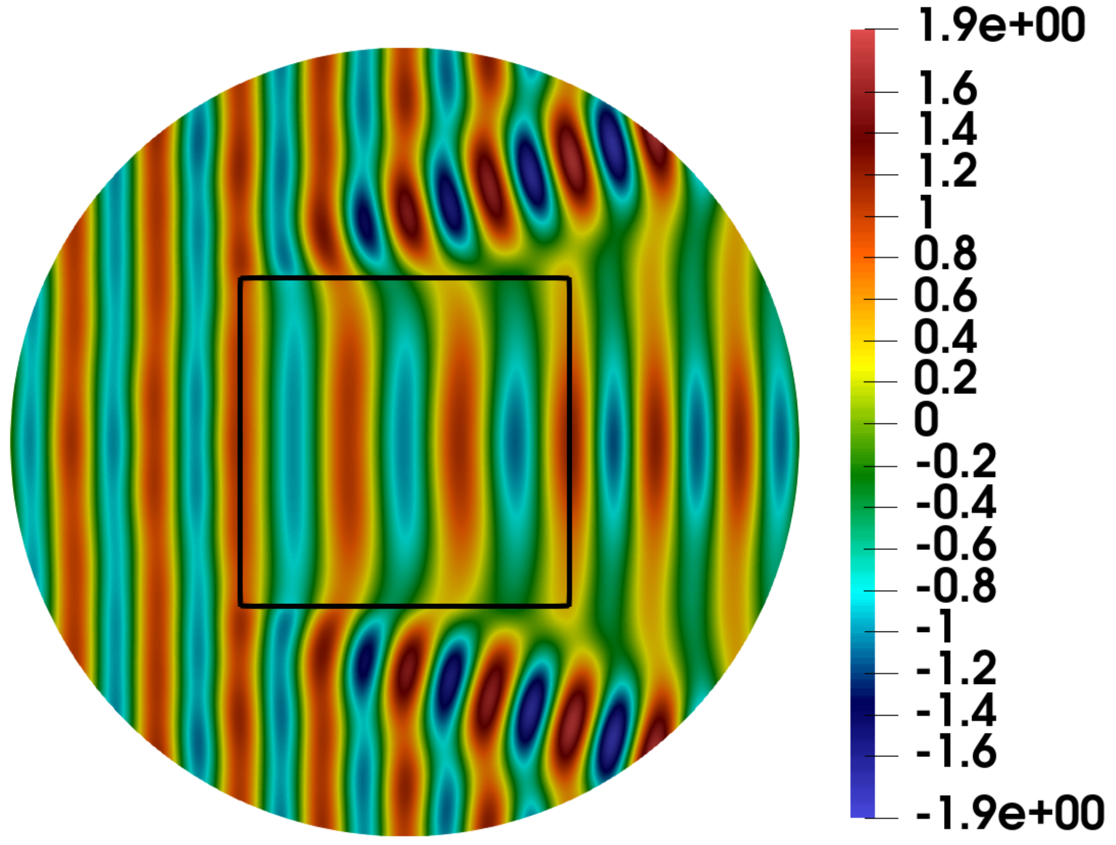
Example of domain



Total field u^{ε}

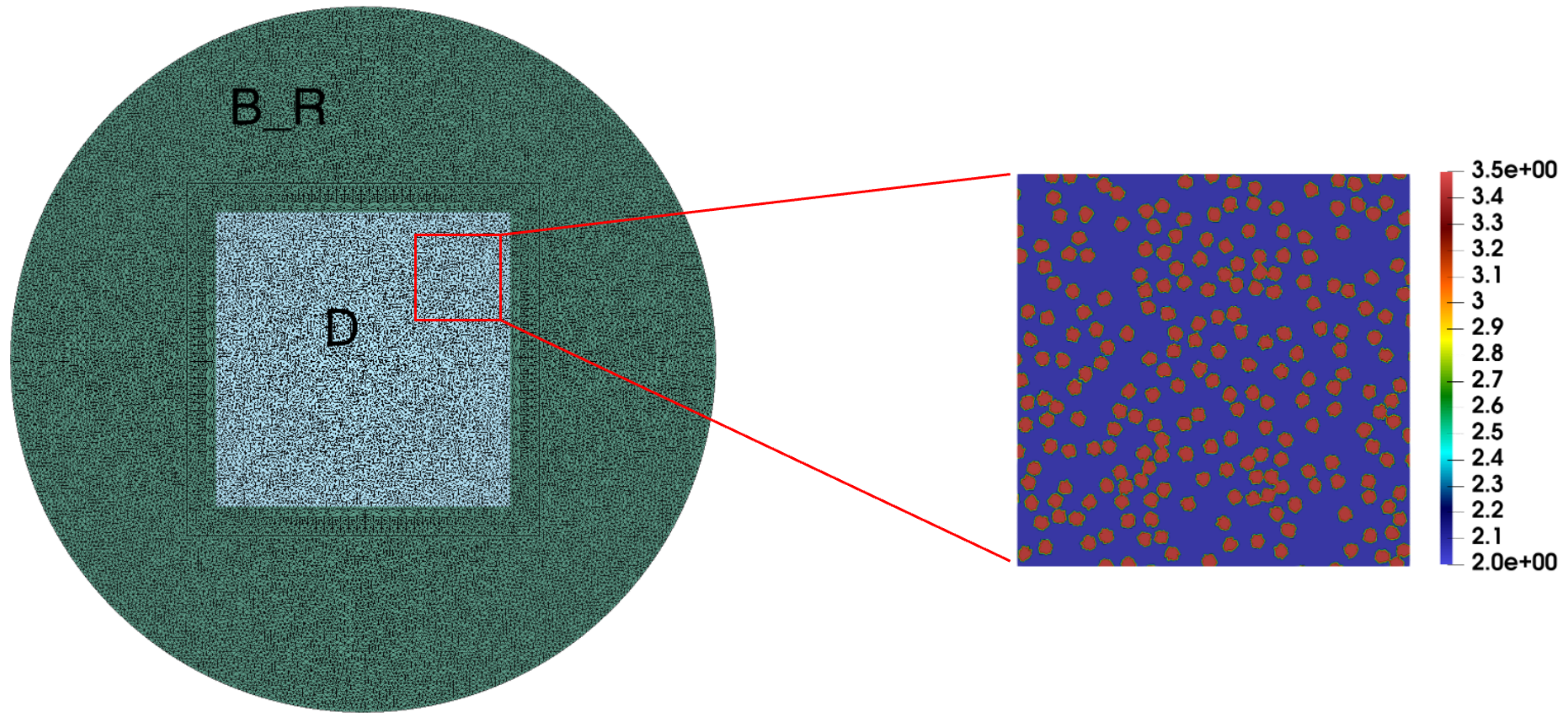


Coherent field u_0

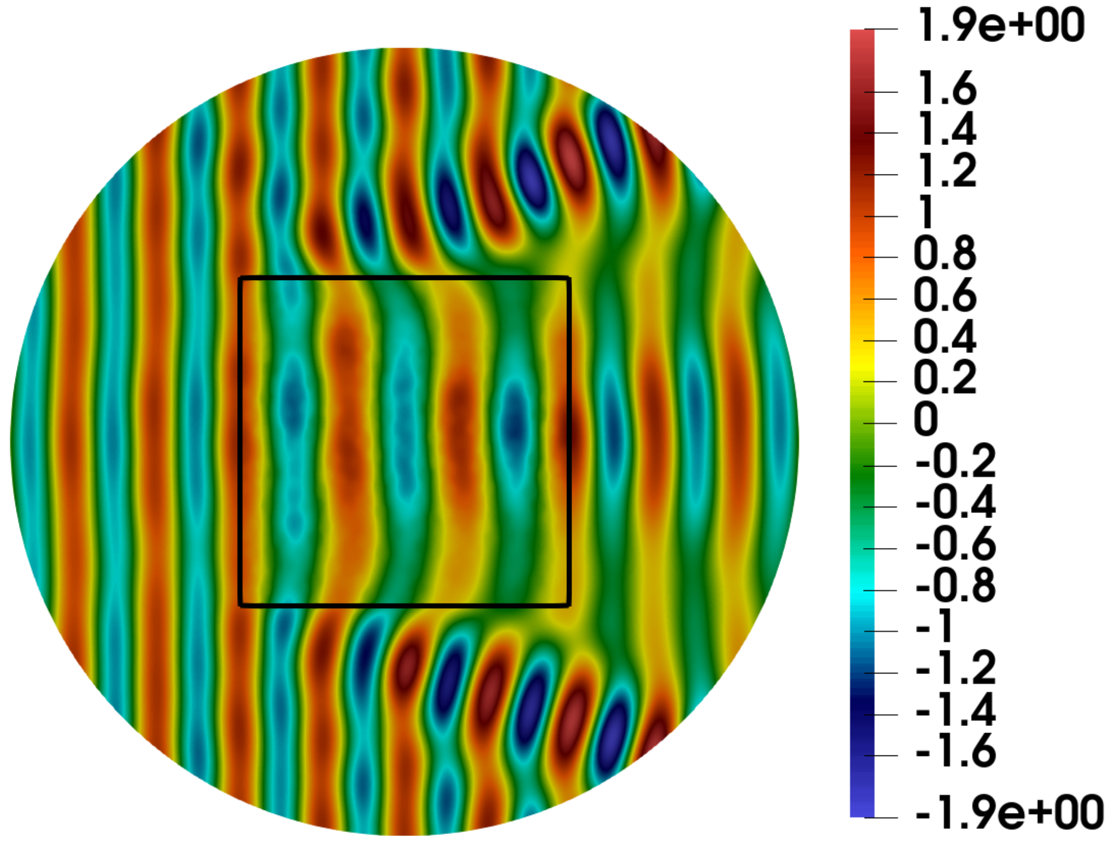


Illustrations

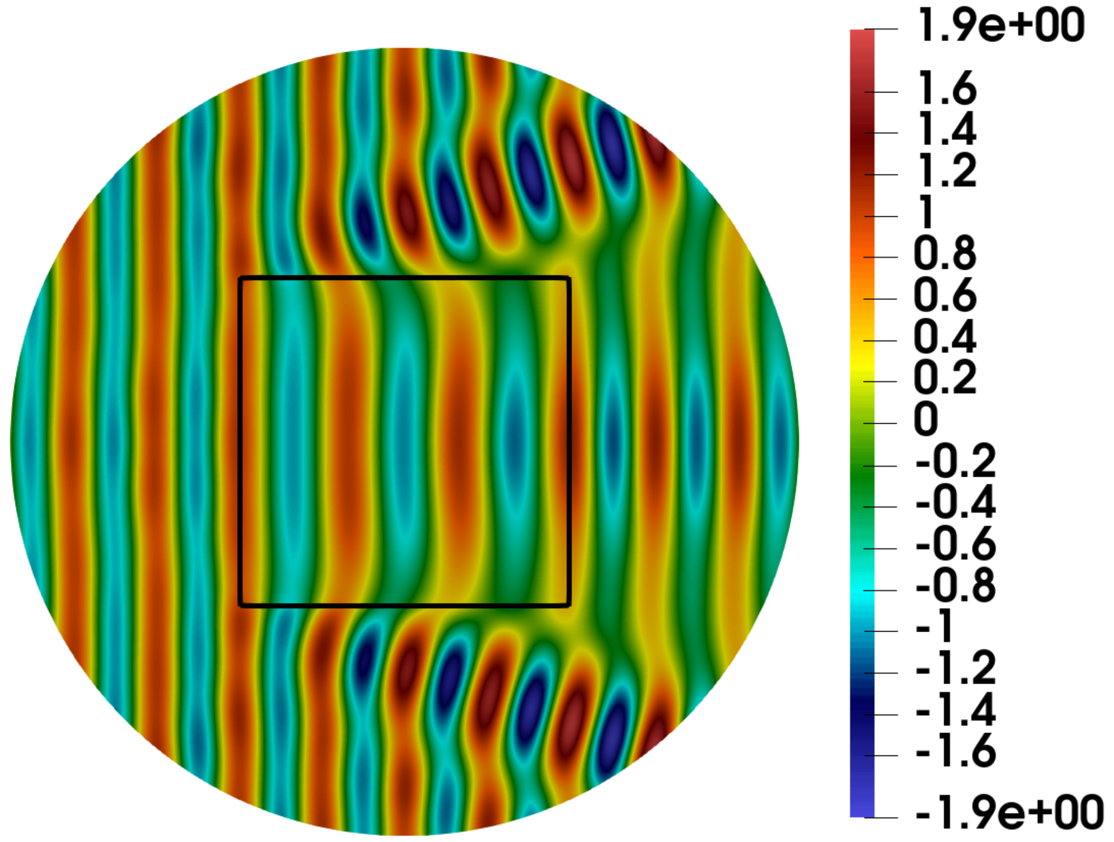
Example of domain



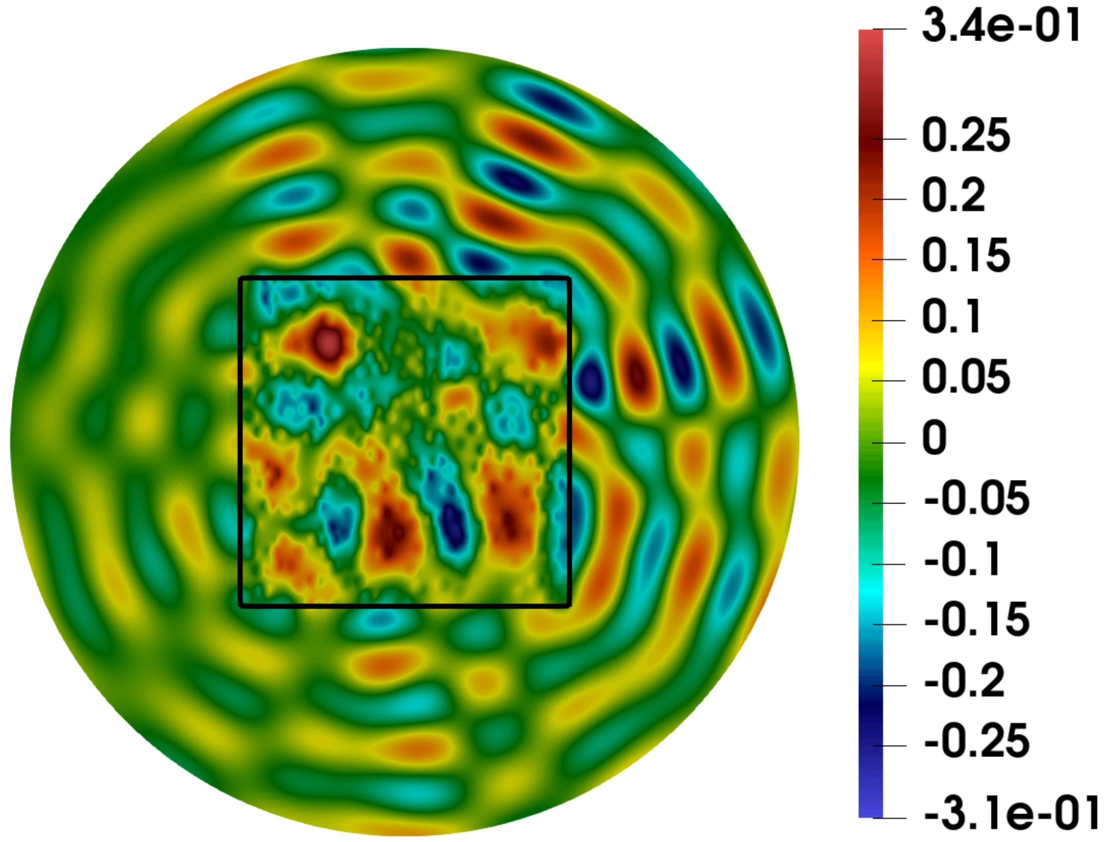
Total field u^ε



Coherent field u_0

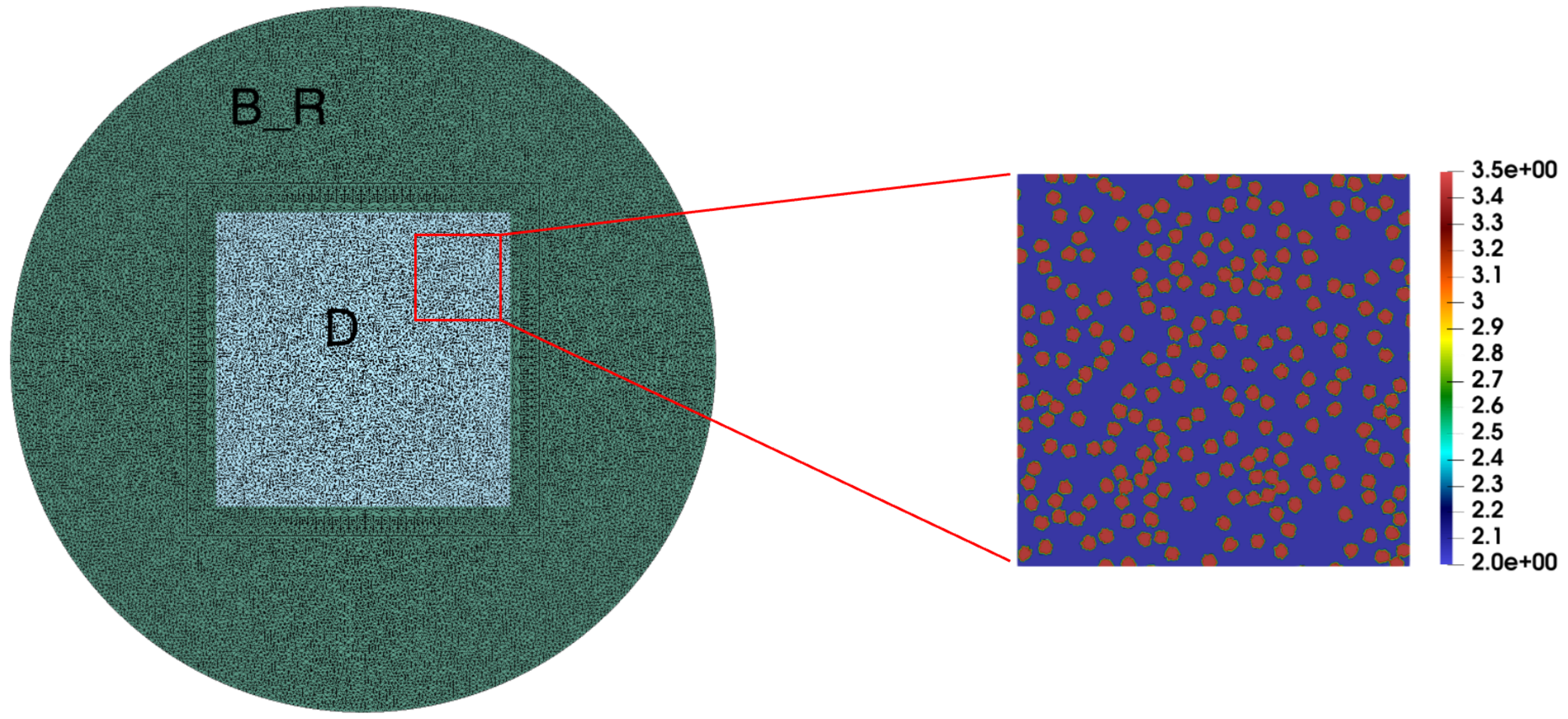


« *Incoherent* » *field* $u^\varepsilon - u_0$

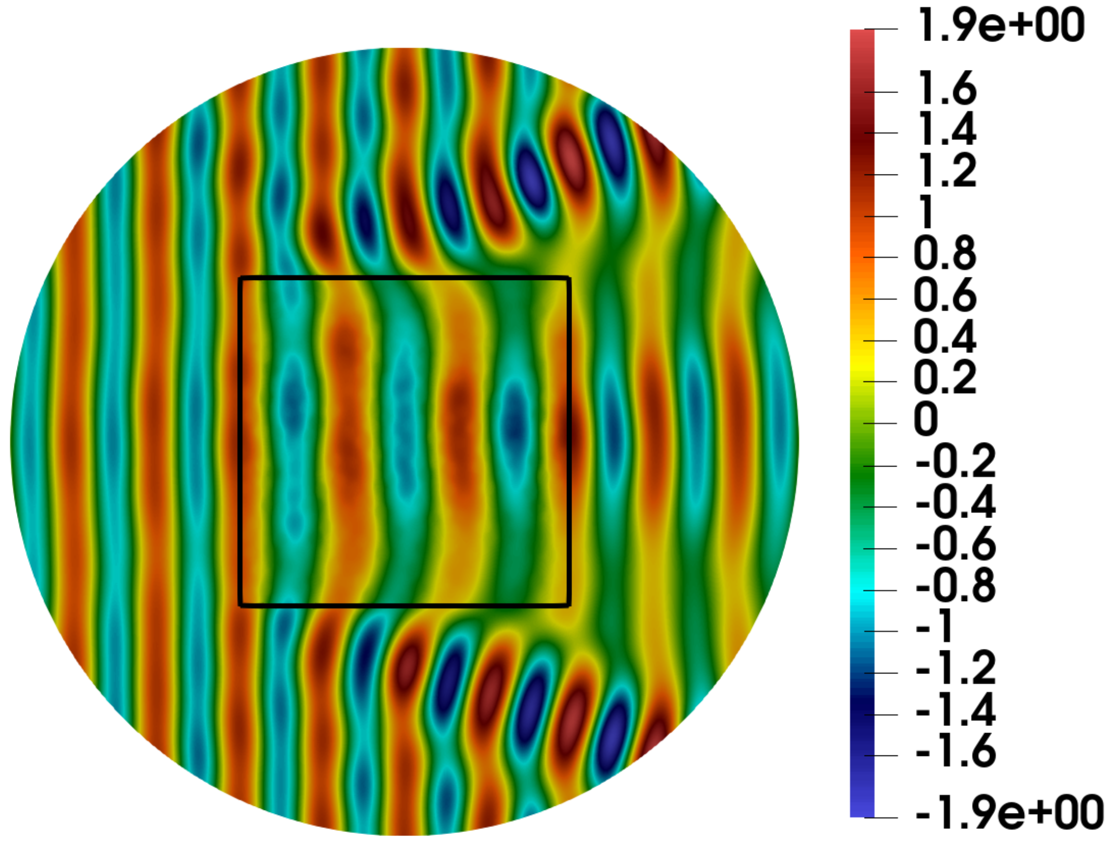


Illustrations

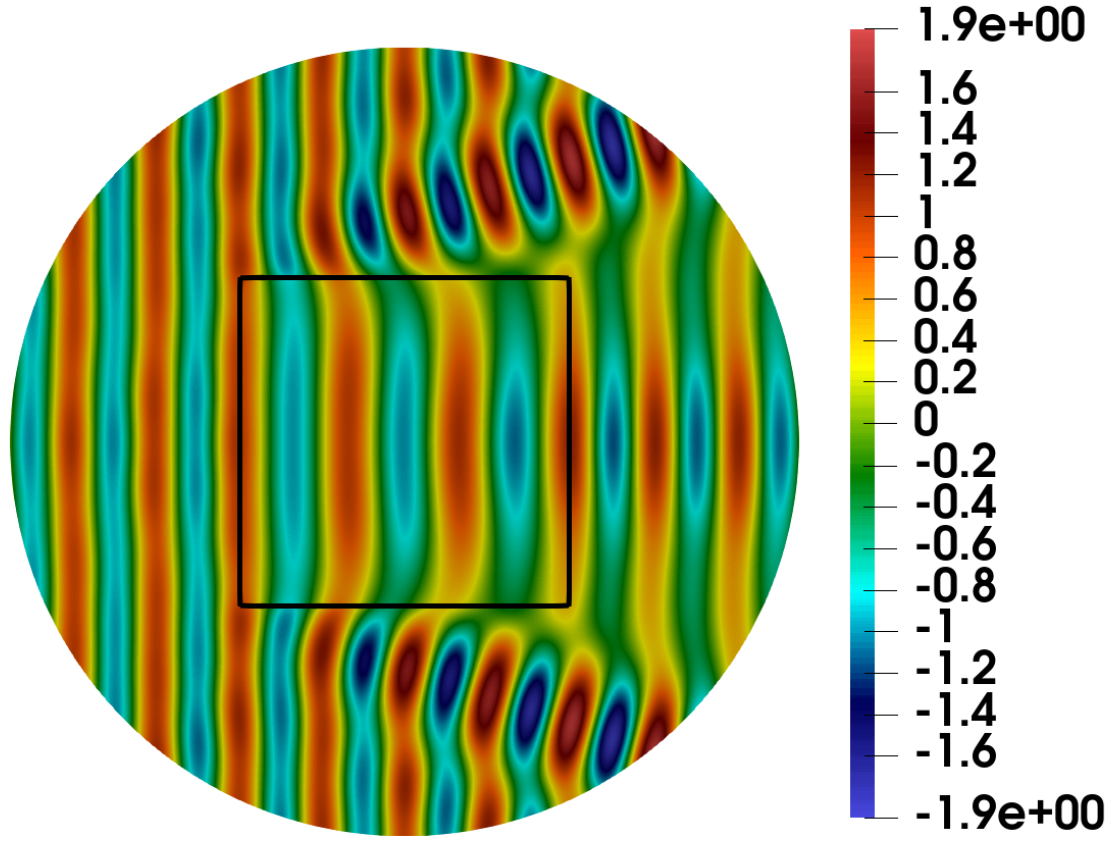
Example of domain



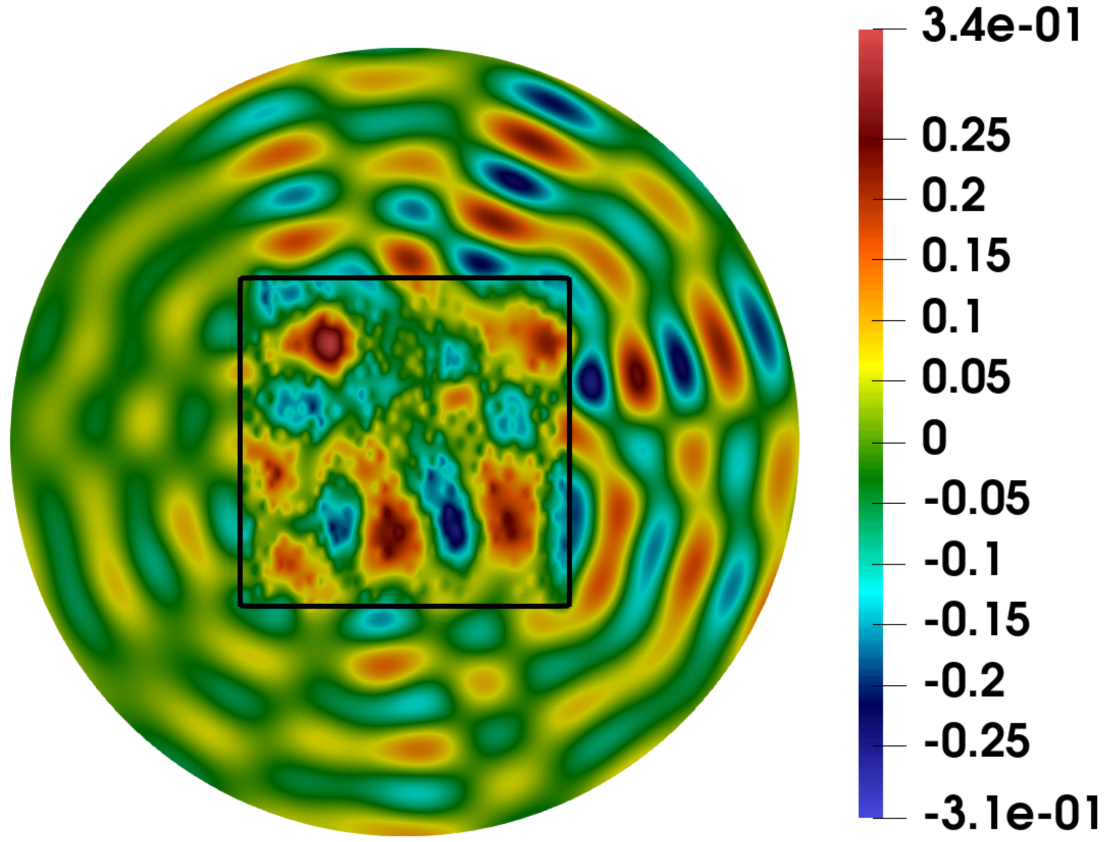
Total field u^ε



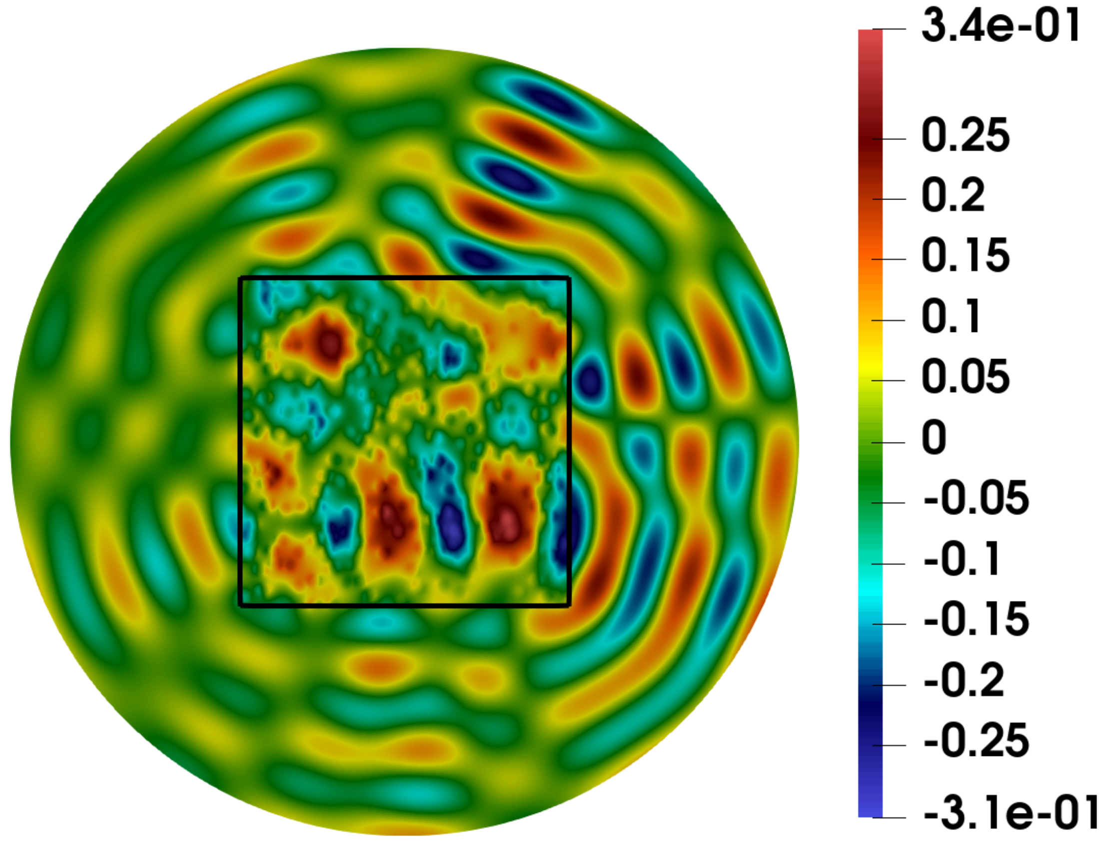
Coherent field u_0



« Incoherent » field $u^\varepsilon - u_0$



$\mathscr{U}^s$ *(First order approximation)*



Conclusion of part 1

It is possible to derive effective properties for the ultrasound propagation from the statistical properties of the micro-structure

It is possible to approximate the backscattered field as a function of the micro-structure and the coherent field

Content

Simple model

Governing equation

$$\begin{cases} \Delta u + \dfrac{\omega^2}{(c^\star)^2} \left(1 + (n_\varepsilon - 1)\, \mathbb{1}_D\right) u = 0 \\ \lim\limits_{|x| \to \infty} |x|^{\frac{d-1}{2}} \left(\partial_{|x|} \left(u - u^i\right) - i \dfrac{\omega}{c^\star} \left(u - u^i\right) \right) = 0. \end{cases}$$

Another simplification for the medium:

Simple model

Governing equation

$$\begin{cases} \Delta u + \dfrac{\omega^2}{(c^\star)^2}\left(1 + (n_\varepsilon - 1)\,\mathbb{1}_D\right) u = 0 \\ \lim\limits_{|x|\to\infty} |x|^{\frac{d-1}{2}} \left(\partial_{|x|}\left(u - u^i\right) - i\dfrac{\omega}{c^\star}\left(u - u^i\right)\right) = 0. \end{cases}$$

Another simplification for the medium:

$$n_\varepsilon(x) = 1 + \tilde{n}\left(\frac{x}{\varepsilon}\right) \text{ with } \mathbb{E}\left[\tilde{n}(x)\right] = 0,$$

Simple model

Governing equation

$$\begin{cases} \Delta u + \dfrac{\omega^2}{(c^\star)^2} \left(1 + (n_\varepsilon - 1)\, \mathbb{1}_D\right) u = 0 \\ \lim\limits_{|x|\to\infty} |x|^{\frac{d-1}{2}} \left(\partial_{|x|} \left(u - u^i\right) - i\dfrac{\omega}{c^\star} \left(u - u^i\right)\right) = 0. \end{cases}$$

Another simplification for the medium:

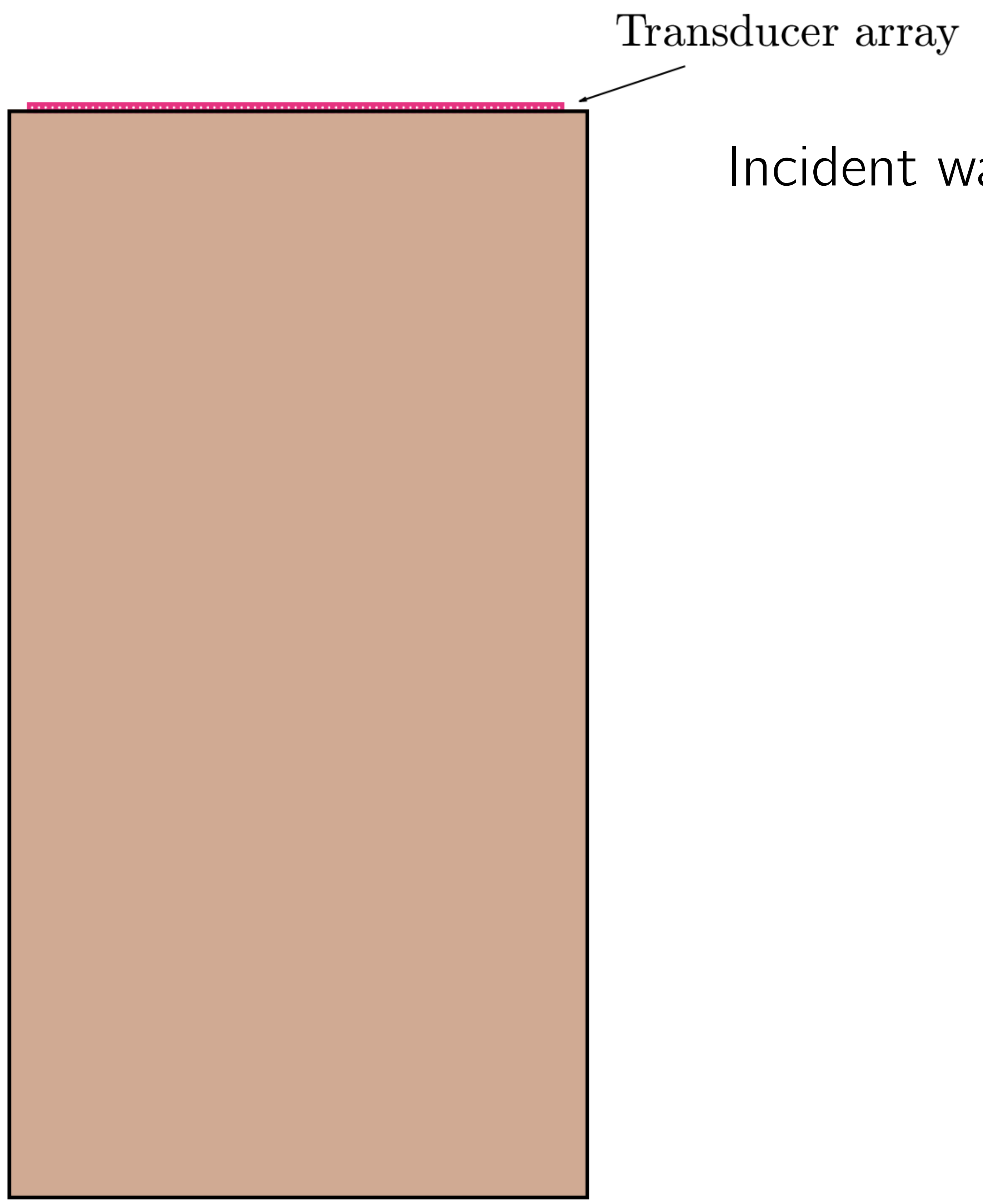
$$n_\varepsilon(x) = 1 + \tilde{n}\left(\frac{x}{\varepsilon}\right) \text{ with } \mathbb{E}\left[\tilde{n}(x)\right] = 0,$$

« **non-restrictive** », just technical simplification to get :

Match between homogenized and outer medium

$$G^\star(\cdot, z) = \Gamma^{\frac{\omega}{c^\star}}(\cdot, z) \text{ and } u^i = u_0.$$

Measurement matrix and imaging functional

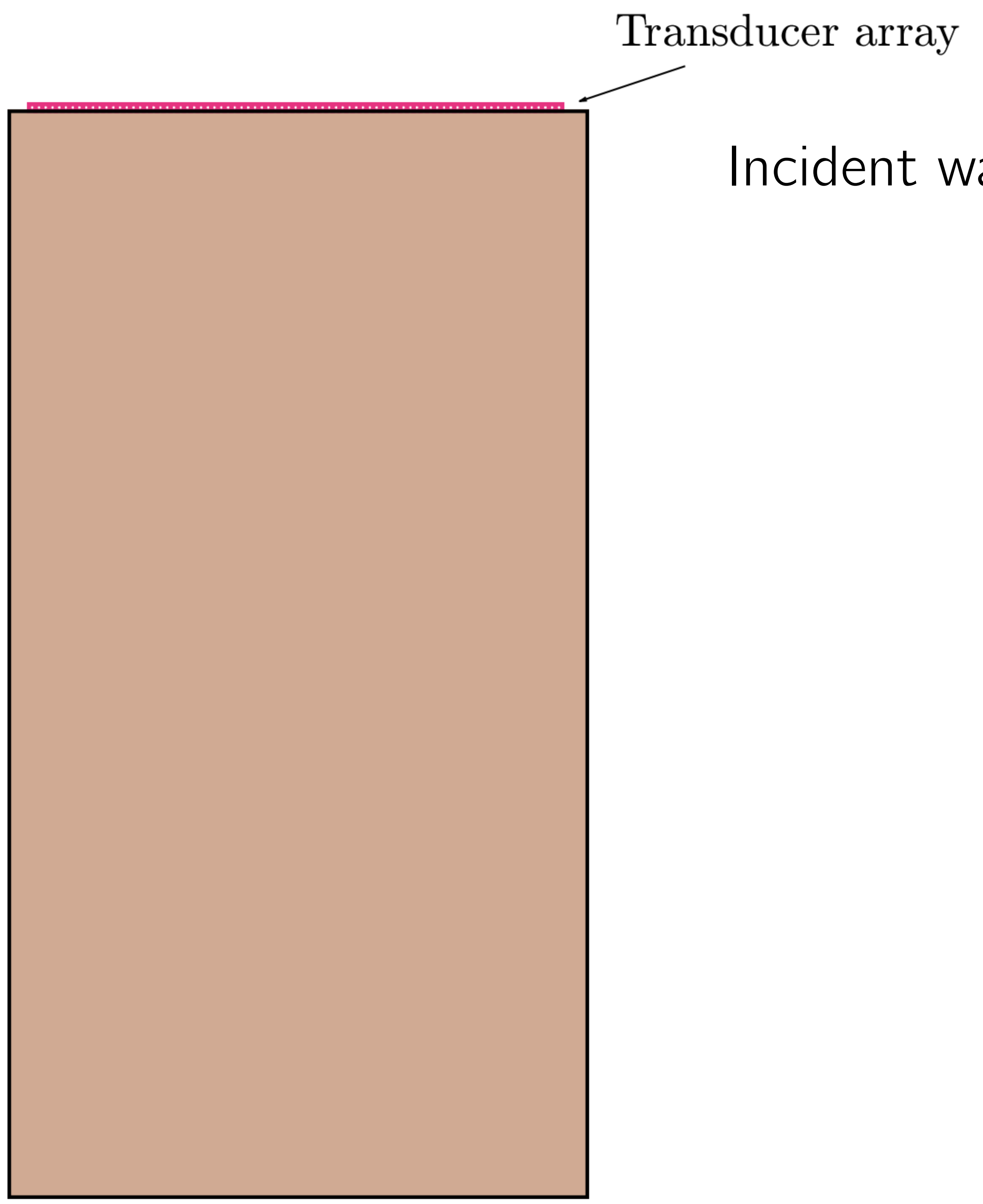


Simulation of US experiment

Physical domain D

Incident wave : $u^i(x_e, \cdot, \omega) := \Gamma^{\frac{\omega}{c^\star}}(x_e, \cdot)$

Measurement matrix and imaging functional

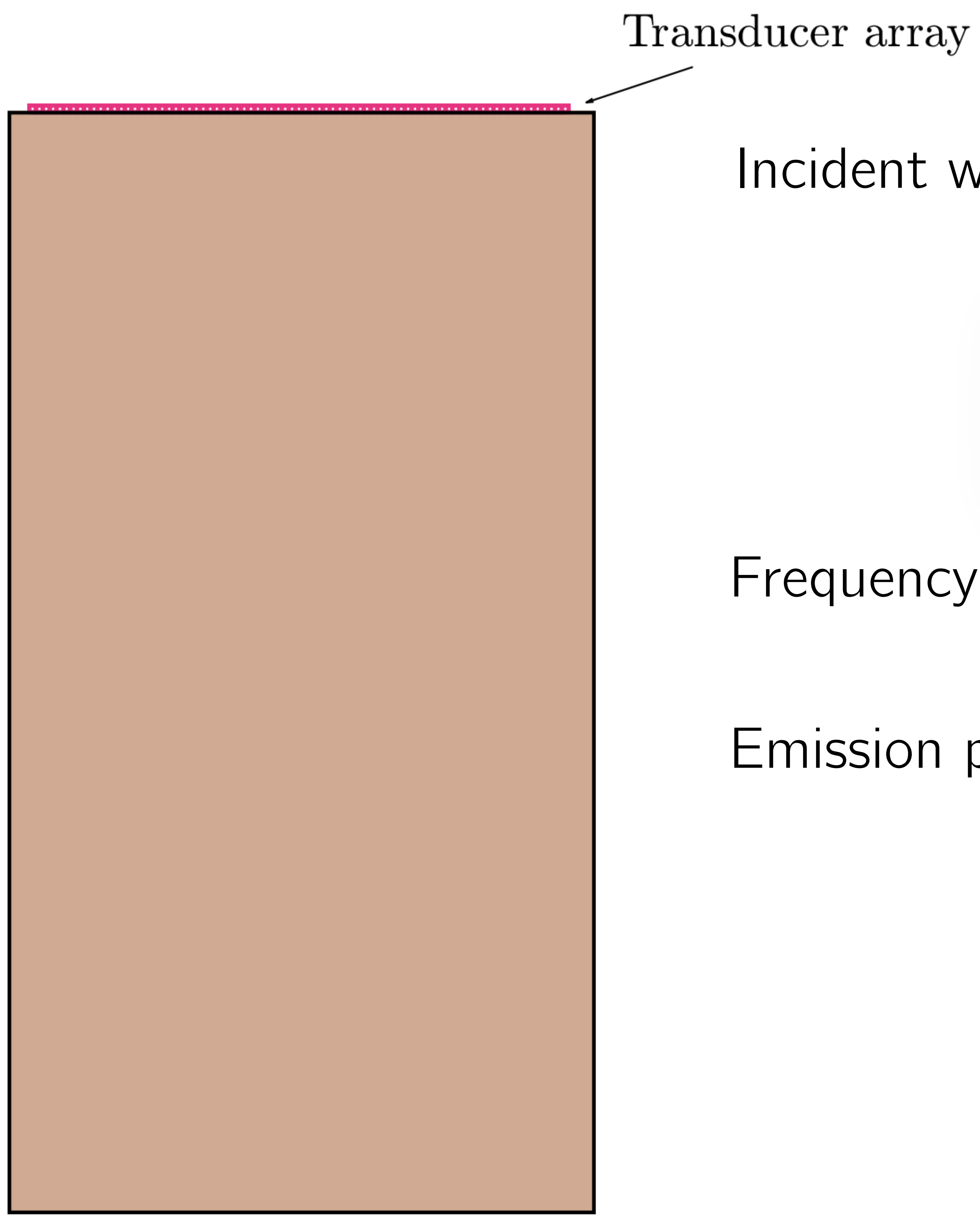


Simulation of US experiment

Physical domain D

Incident wave : $u^i(x_e, \cdot, \omega) := \Gamma^{\frac{\omega}{c^\star}}(x_e, \cdot)$

Measurement matrix and imaging functional



Simulation of US experiment

Physical domain D

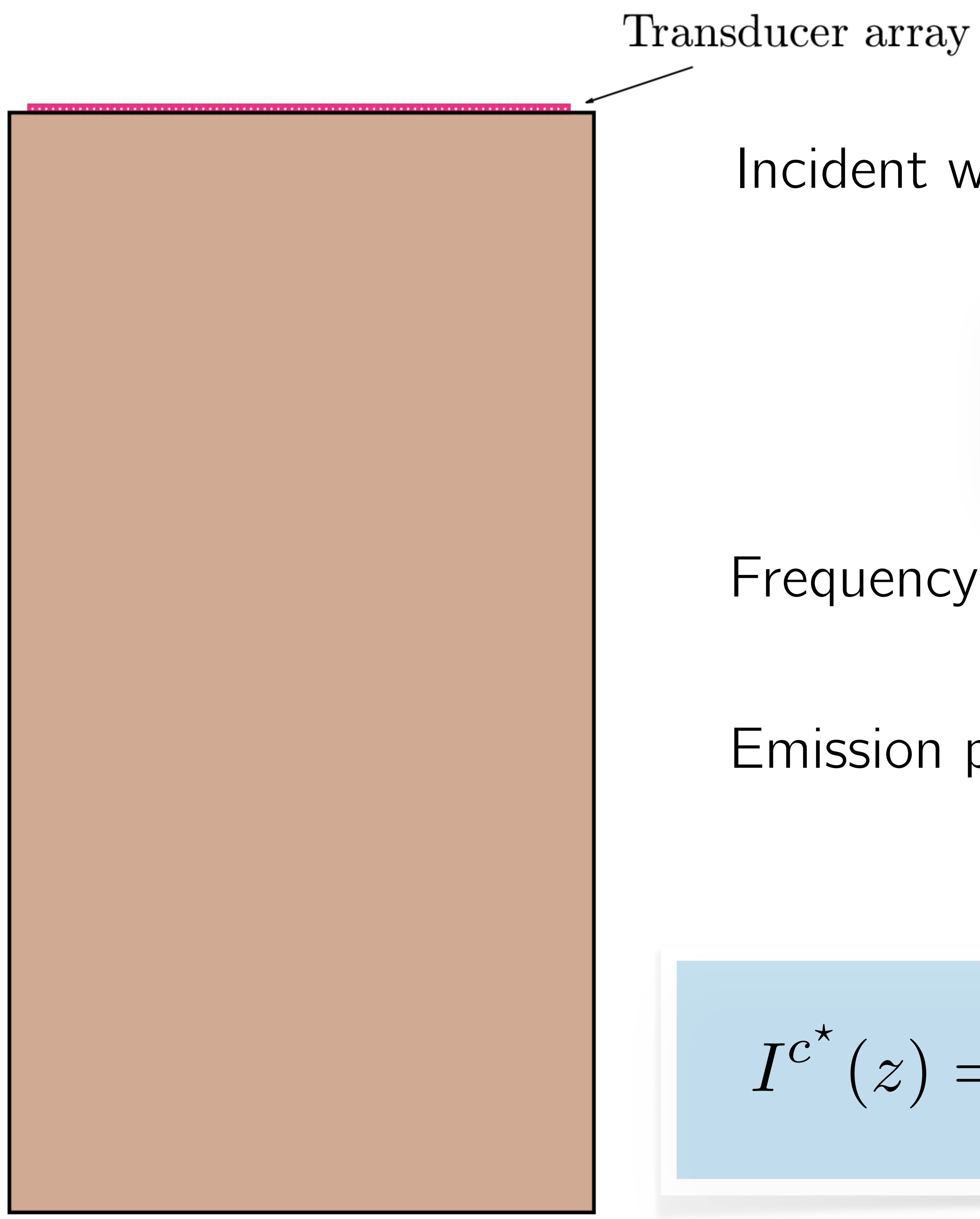
Incident wave : $u^i(x_e, \cdot, \omega) := \Gamma^{\frac{\omega}{c^\star}}(x_e, \cdot)$

Measurement matrix: $M_{e,r}(\omega) := u^s(x_e, x_r, \omega)$

Frequency $\omega \in \mathcal{B}$ (bandwidth). Effective speed of sound $c^\star$. Probe $\mathcal{P} = [-\frac{a}{2}, \frac{a}{2}] \times 0,$

Emission point $x_e \in \mathcal{P}$, reception point $x_r \in \mathcal{P}$.

Measurement matrix and imaging functional



Simulation of US experiment

Physical domain D

Incident wave : $u^i(x_e, \cdot, \omega) := \Gamma^{\frac{\omega}{c^\star}}(x_e, \cdot)$

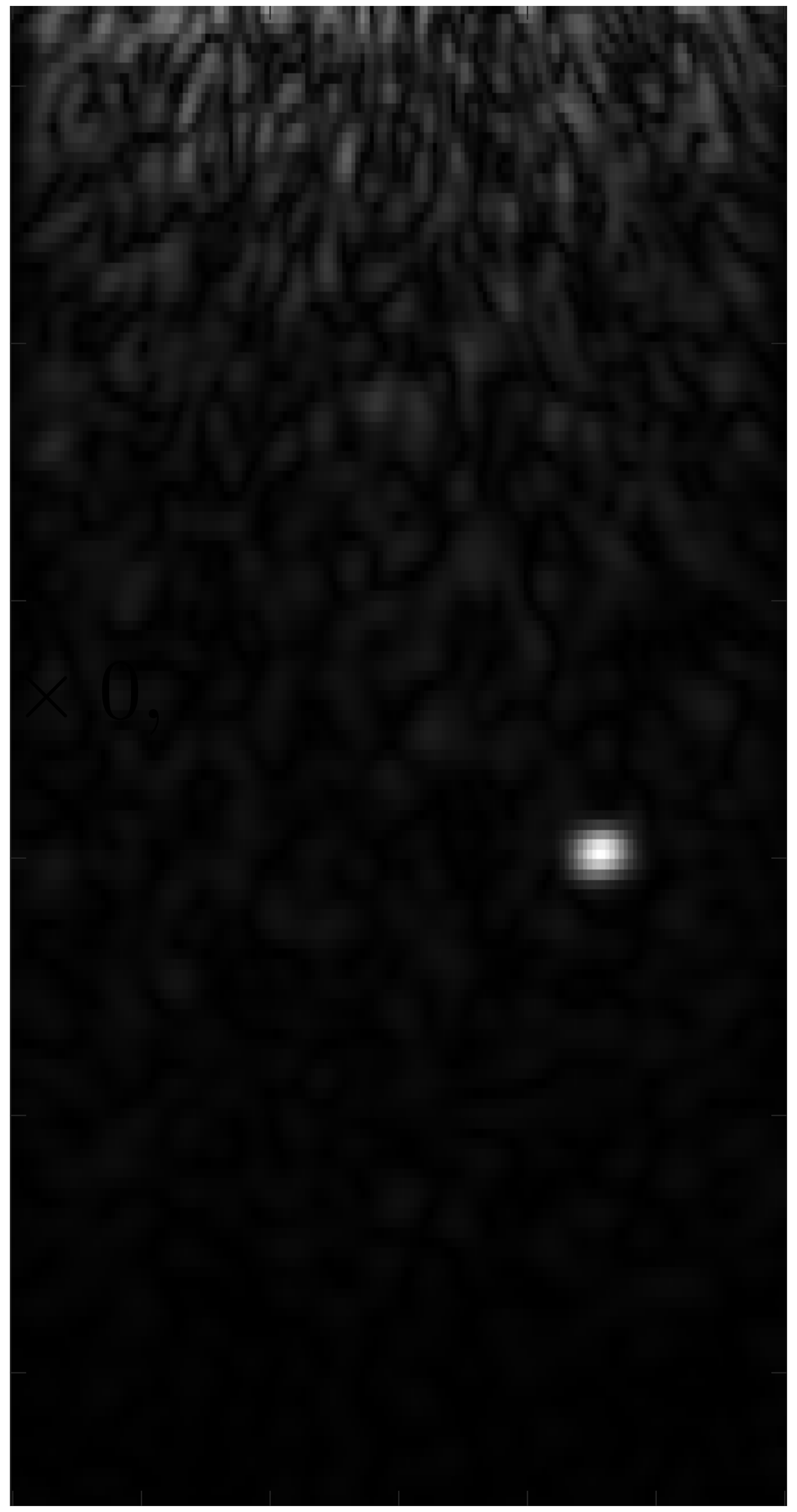
Measurement matrix: $M_{e,r}(\omega) := u^s(x_e, x_r, \omega)$

Frequency $\omega \in \mathcal{B}$ (bandwidth). Effective speed of sound $c^\star$. Probe $\mathcal{P} = [-\frac{a}{2}, \frac{a}{2}] \times 0$.

Emission point $x_e \in \mathcal{P}$, reception point $x_r \in \mathcal{P}$.

Imaging functional

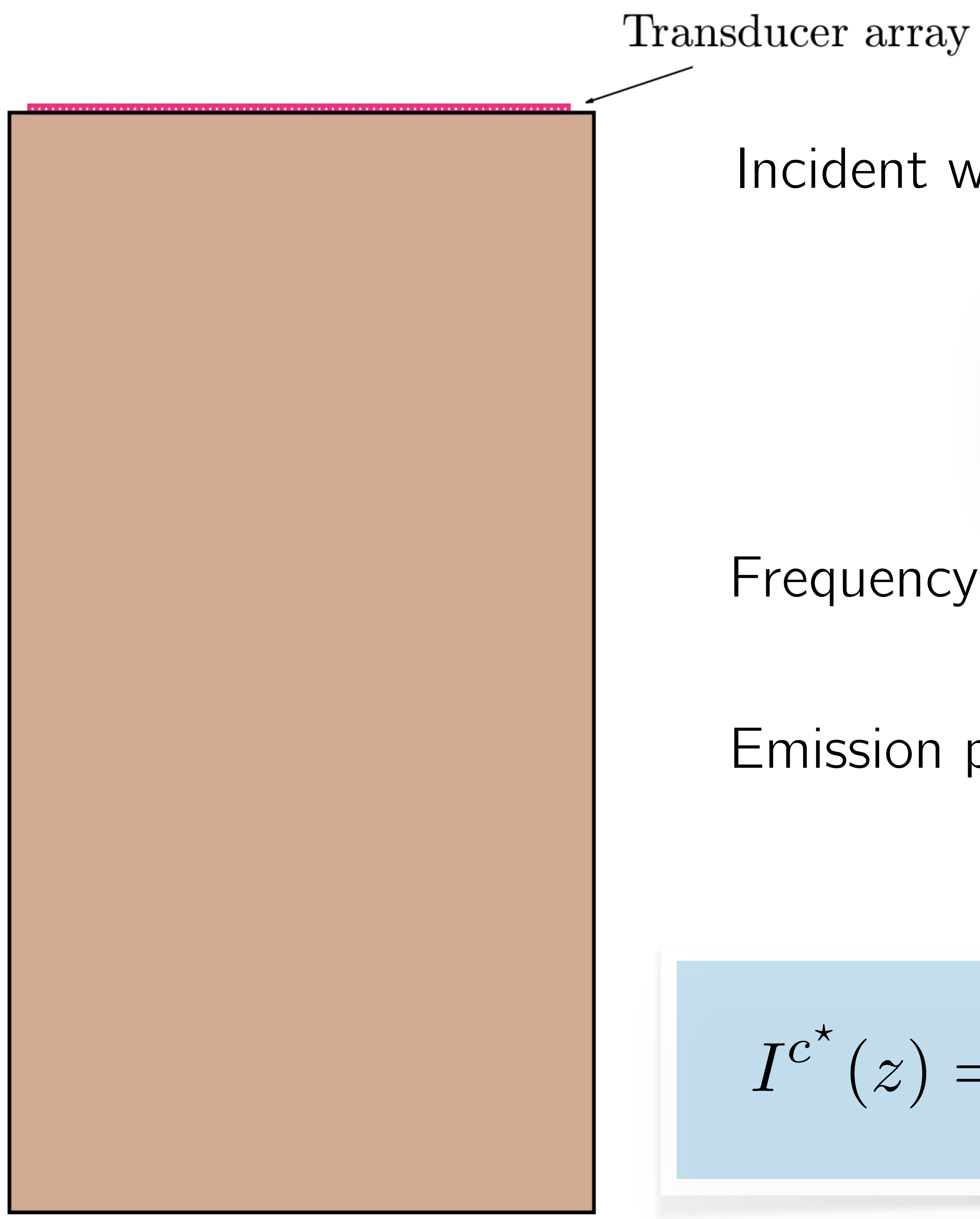
$$I^{c^\star}(z) = \int_{\mathcal{A}\times\mathcal{A}\times\mathcal{B}} \overline{u^s(x_e, x_r, \omega)}\Gamma^{\frac{\omega}{c^\star}}(x_e, z)\,\Gamma^{\frac{\omega}{c^\star}}(x_r, z)\,\mathrm{d}x_e \mathrm{d}x_r \mathrm{d}\omega$$



Computed Image

Imaging domain D'

Measurement matrix and imaging functional



Incident wave : $u^i(x_e,\cdot,\omega) := \Gamma^{\frac{\omega}{c^\star}}(x_e,\cdot)$

Measurement matrix: $M_{e,r}(\omega) := u^s(x_e,x_r,\omega)$

Frequency $\omega \in \mathcal{B}$ (bandwidth). Effective speed of sound $c^\star$. Probe $\mathcal{P} = [-\frac{a}{2},\frac{a}{2}] \times 0$.

Emission point $x_e \in \mathcal{P}$, reception point $x_r \in \mathcal{P}$.

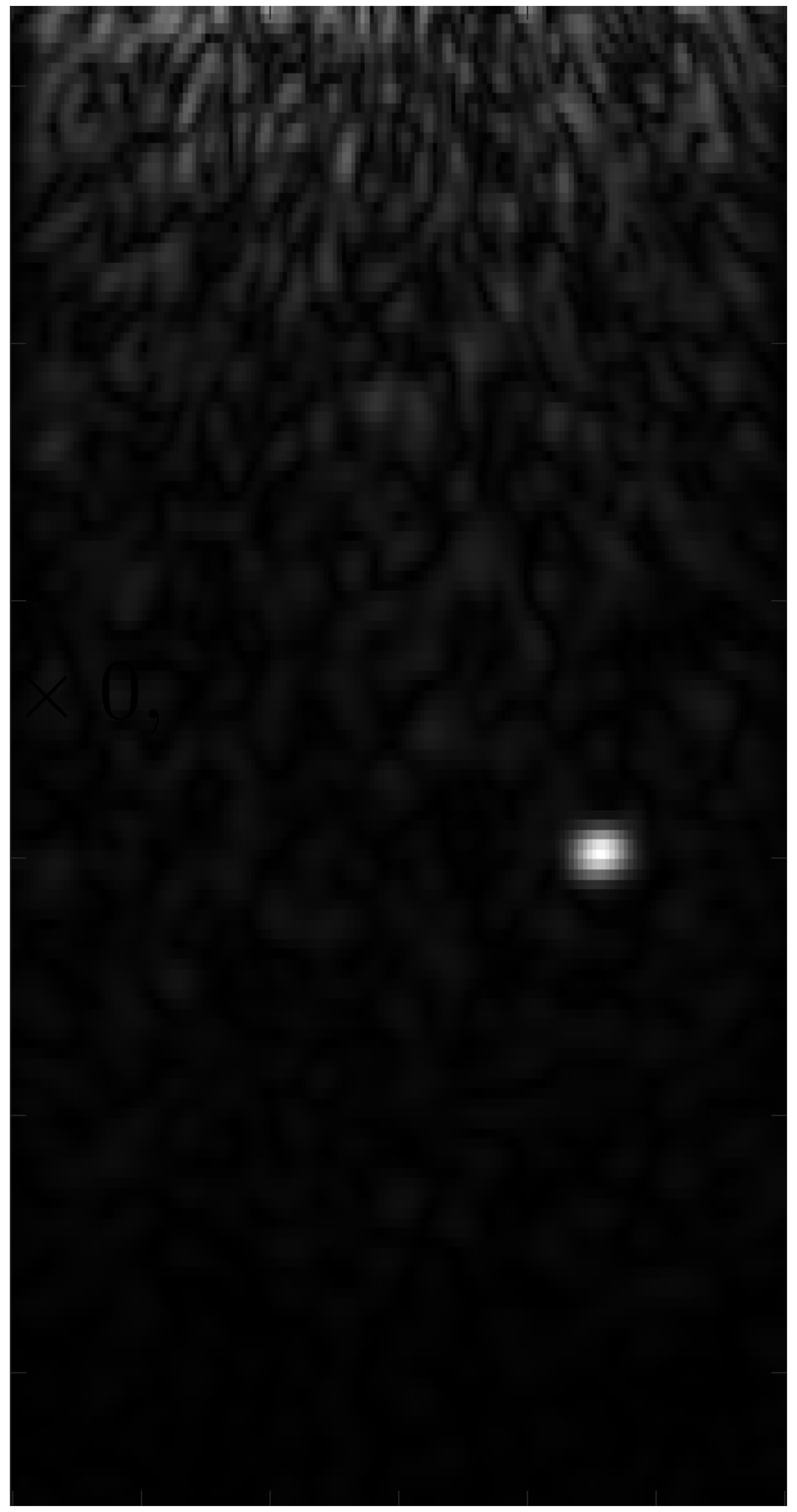
Imaging functional

$$I^{c^\star}(z) = \int_{\mathcal{A}\times\mathcal{A}\times\mathcal{B}} \overline{u^s(x_e,x_r,\omega)}\Gamma^{\frac{\omega}{c^\star}}(x_e,z)\,\Gamma^{\frac{\omega}{c^\star}}(x_r,z)\,\mathrm{d}x_e\mathrm{d}x_r\mathrm{d}\omega$$

What if $c^\star$ is not known ?

Simulation of US experiment

Physical domain D



Computed Image

Imaging domain D'

Characterization of the problem

Geometry

Active array of sources / sensor outside the medium

Reflection geometry

Large domain

Regime

Random fluctuations of the medium:
short correlation length & order 1 amplitude

Dense distribution of scatterers (no sparsity)

Goal

Find the effective speed of sound *in situ*
(geometrical component of the focusing law)

Constraints

Low computational cost

Prior

Stationarity and ergodicity of the fluctuations

The point-spread function (PSF)

First order approximation for the scattered fields

$$M(x_e, x_r, \omega) = \omega^2 \int_D (n(y) - n^\star)\, G^\star(x_r, y) G^\star(x_e, y) \mathrm{d}y$$

The point-spread function (PSF)

First order approximation for the scattered fields

$$M(x_e, x_r, \omega) = \omega^2 \int_D (n(y) - n^\star)\, G^\star(x_r, y) G^\star(x_e, y) \mathrm{d}y$$

Imaging functional as a convolution

$$I^\star(z) = \int_D (n(y) - n^\star)\, F^{c^\star}(z, y) \mathrm{d}y, \qquad z \in D'$$

The point-spread function (PSF)

First order approximation for the scattered fields

$$M(x_e, x_r, \omega) = \omega^2 \int_D (n(y) - n^\star)\, G^\star(x_r, y) G^\star(x_e, y) \mathrm{d}y$$

Imaging functional as a convolution

$$I^\star(z) = \int_D (n(y) - n^\star)\, F^{c^\star}(z, y) \mathrm{d}y, \qquad z \in D'$$

Kernel of the operator (Point Spread Function)

$$F^\star(z, y) = \int_{\mathcal{B}} \omega^2 \left(\int_{\mathcal{P}} \Gamma^{\frac{\omega}{c^\star}}(z, x_r) \overline{G^\star(y, x_r)} \mathrm{d}x_r \right)^2 \mathrm{d}\omega \qquad z, y \in D' \times D.$$

The case $c \neq c^\star$

First order approximation for the scattered fields

$$M(x_e, x_r, \omega) = \omega^2 \int_D (n(y) - n^\star)\, G^\star(x_r, y) G^\star(x_e, y) \mathrm{d}y$$

(Unchanged)

The case $c \neq c^\star$

First order approximation for the scattered fields

$$M(x_e, x_r, \omega) = \omega^2 \int_D (n(y) - n^\star)\, G^\star(x_r, y) G^\star(x_e, y) \mathrm{d}y$$

(Unchanged)

Imaging functional as a convolution

$$I^c(z) = \int_D (n(y) - n^\star)\, F^{\frac{c}{c^\star}}(z, y) \mathrm{d}y, \qquad z \in D'$$

The case $c \neq c^\star$

First order approximation for the scattered fields

$$M(x_e, x_r, \omega) = \omega^2 \int_D (n(y) - n^\star)\, G^\star(x_r, y) G^\star(x_e, y) \mathrm{d}y$$

(Unchanged)

Imaging functional as a convolution

$$I^c(z) = \int_D (n(y) - n^\star)\, F^{\frac{c}{c^\star}}(z, y) \mathrm{d}y, \qquad z \in D'$$

$D' \leftarrow \varphi_c(D)$

The case $c \neq c^\star$

First order approximation for the scattered fields

$$M(x_e, x_r, \omega) = \omega^2 \int_D (n(y) - n^\star)\, G^\star(x_r, y) G^\star(x_e, y) \mathrm{d}y$$

(Unchanged)

Imaging functional as a convolution

$$I^c(z) = \int_D (n(y) - n^\star)\, F^{\frac{c}{c^\star}}(z, y) \mathrm{d}y, \qquad z \in D'$$

$D' \leftarrow \varphi_c(D)$

Kernel of the operator (Point Spread Function)

$$F^{\frac{c}{c^\star}}(z, y) = \int_{\mathcal{B}} \omega^2 \left(\int_{\mathcal{P}} \Gamma^{\frac{\omega}{c}}(z, x_r) \overline{G^\star(y, x_r)} \mathrm{d}x_r \right)^2 \mathrm{d}\omega \qquad z, y \in D' \times D.$$

Effects of speed of sound error on the focal spot

$$\begin{aligned} D' &\longrightarrow D' \\ z &\longmapsto \left| F^{\frac{c}{c^\star}}(z, y_0) \right| \end{aligned}$$

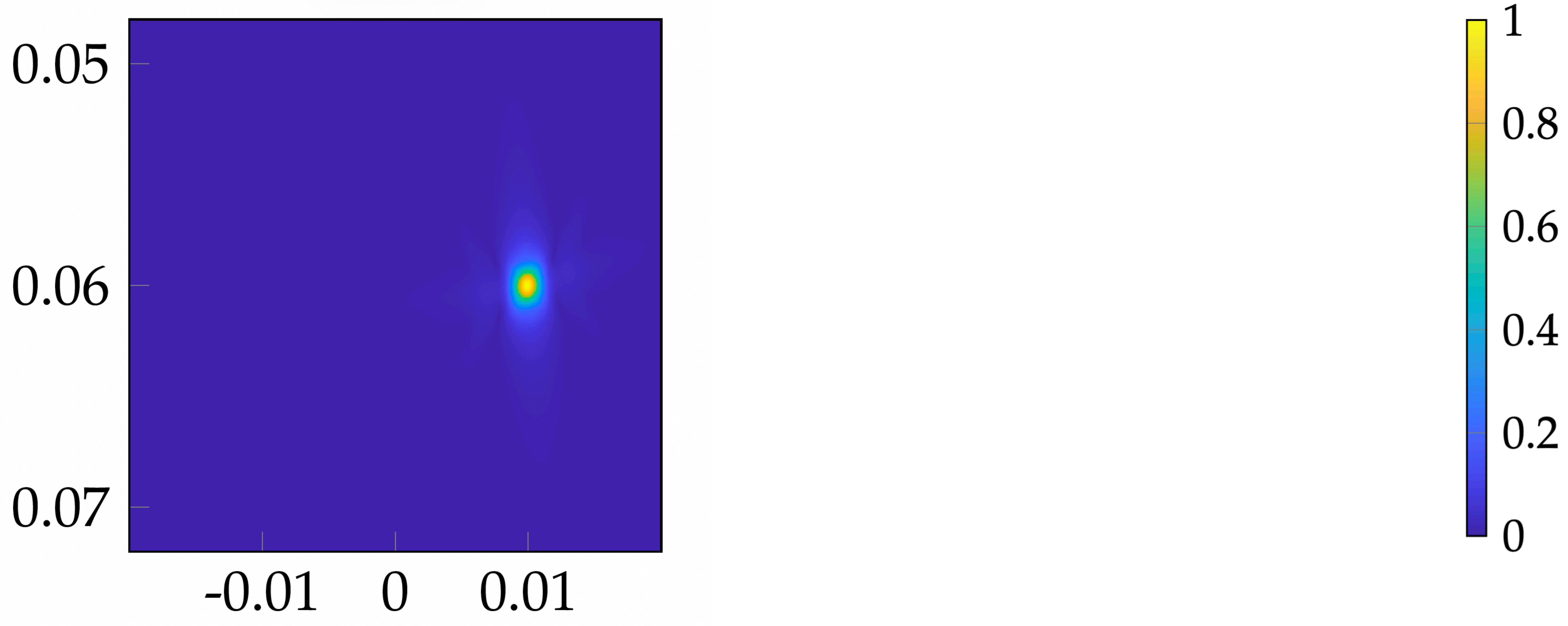
$y_0 = (0.01, 0.06) \in D$

Effects of speed of sound error on the focal spot

$$\begin{aligned} D' &\longrightarrow D' \\ z &\longmapsto \left|F^{\frac{c}{c^\star}}(z, y_0)\right| \end{aligned}$$

$y_0 = (0.01, 0.06) \in D$

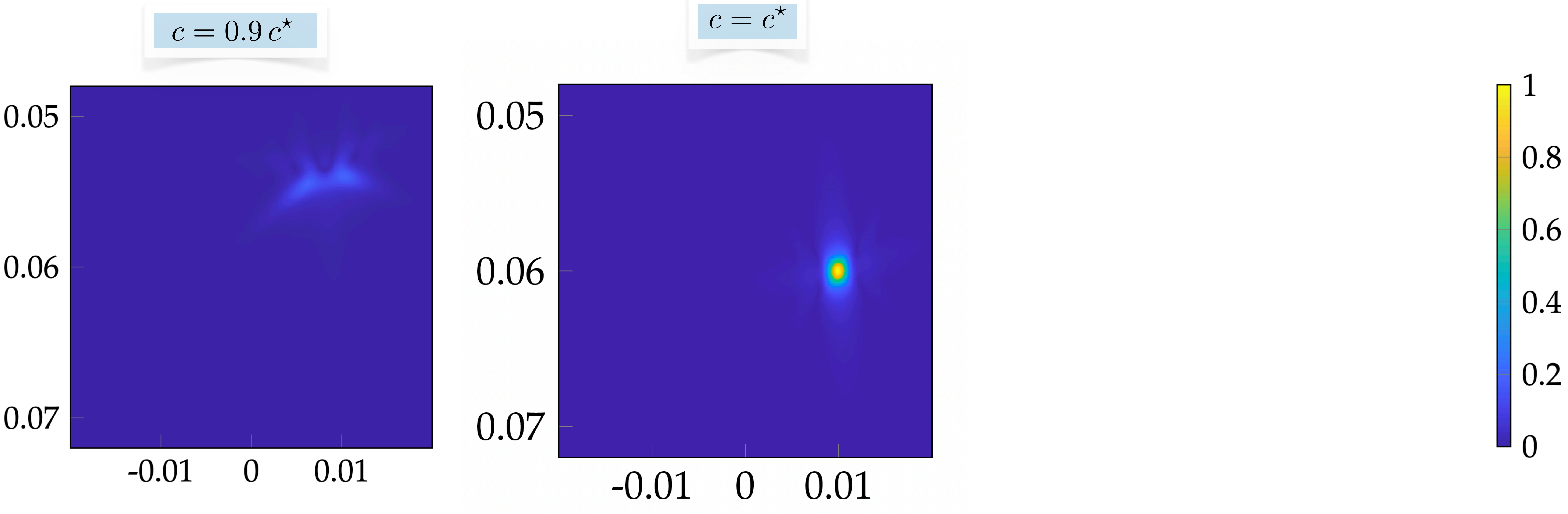
$c = c^\star$



Effects of speed of sound error on the focal spot

$$\begin{aligned} D' &\longrightarrow D' \\ z &\longmapsto \left|F^{\frac{c}{c^\star}}(z, y_0)\right| \end{aligned}$$

$y_0 = (0.01, 0.06) \in D$

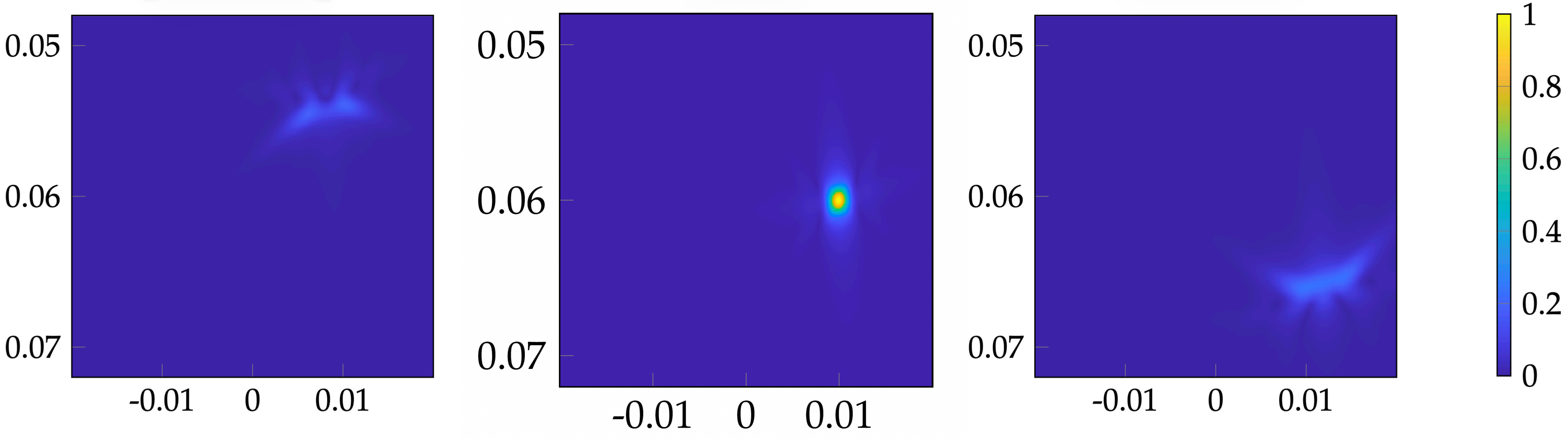


Effects of speed of sound error on the focal spot

$$\begin{aligned} D' &\longrightarrow D' \\ z &\longmapsto \left|F^{\frac{c}{c^\star}}(z, y_0)\right| \end{aligned}$$

$y_0 = (0.01, 0.06) \in D$

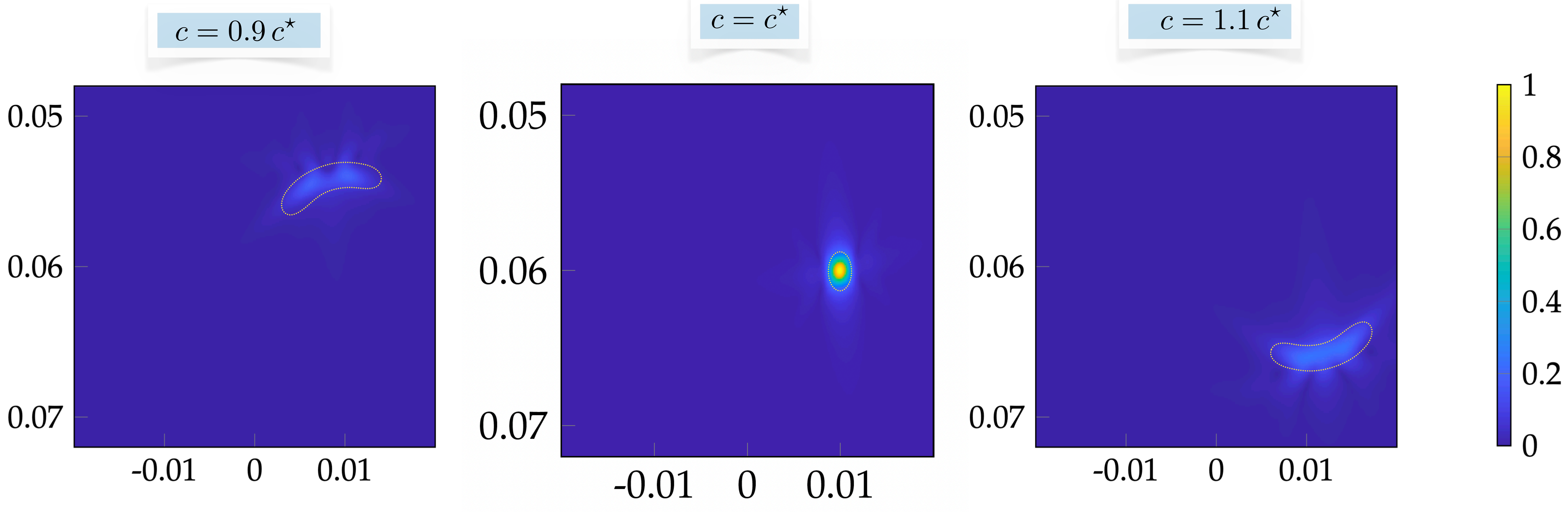
$c = 0.9\,c^\star$ $\quad$ $c = c^\star$ $\quad$ $c = 1.1\,c^\star$



Effects of speed of sound error on the focal spot

$$\begin{aligned} D' &\longrightarrow D' \\ z &\longmapsto \left|F^{\frac{c}{c^\star}}(z, y_0)\right| \end{aligned}$$

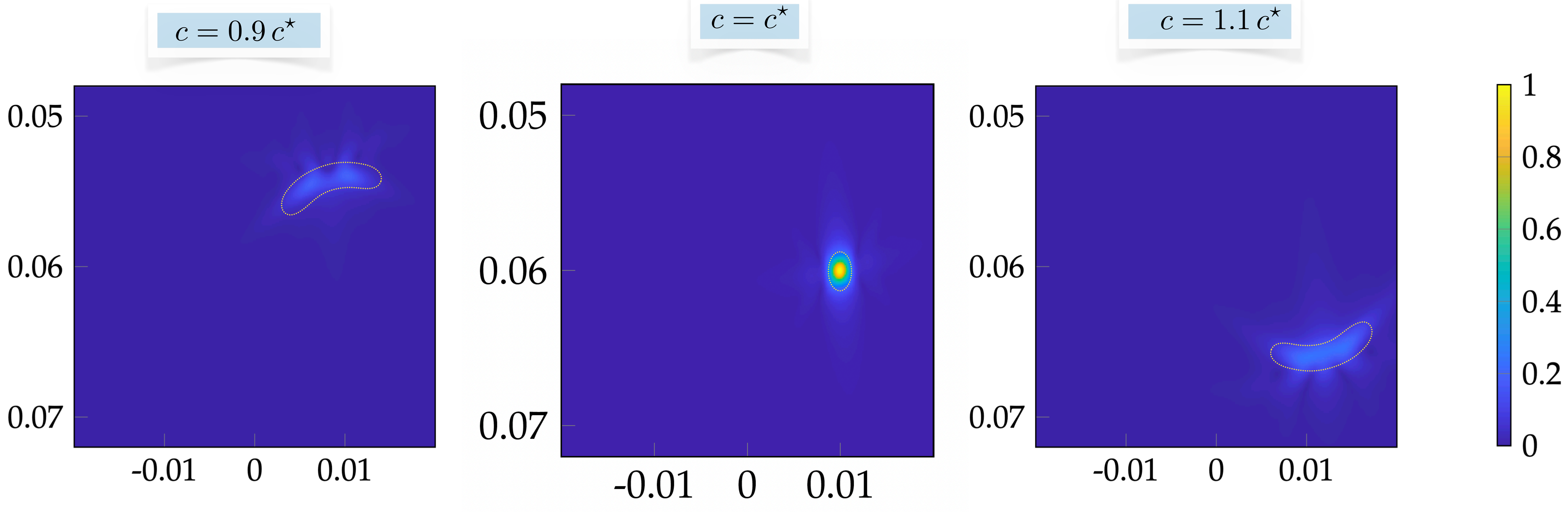
$y_0 = (0.01, 0.06) \in D$



Effects of speed of sound error on the focal spot

$$\begin{aligned} D' &\longrightarrow D' \\ z &\longmapsto \left| F^{\frac{c}{c^\star}}(z, y_0) \right| \end{aligned}$$

$y_0 = (0.01, 0.06) \in D$

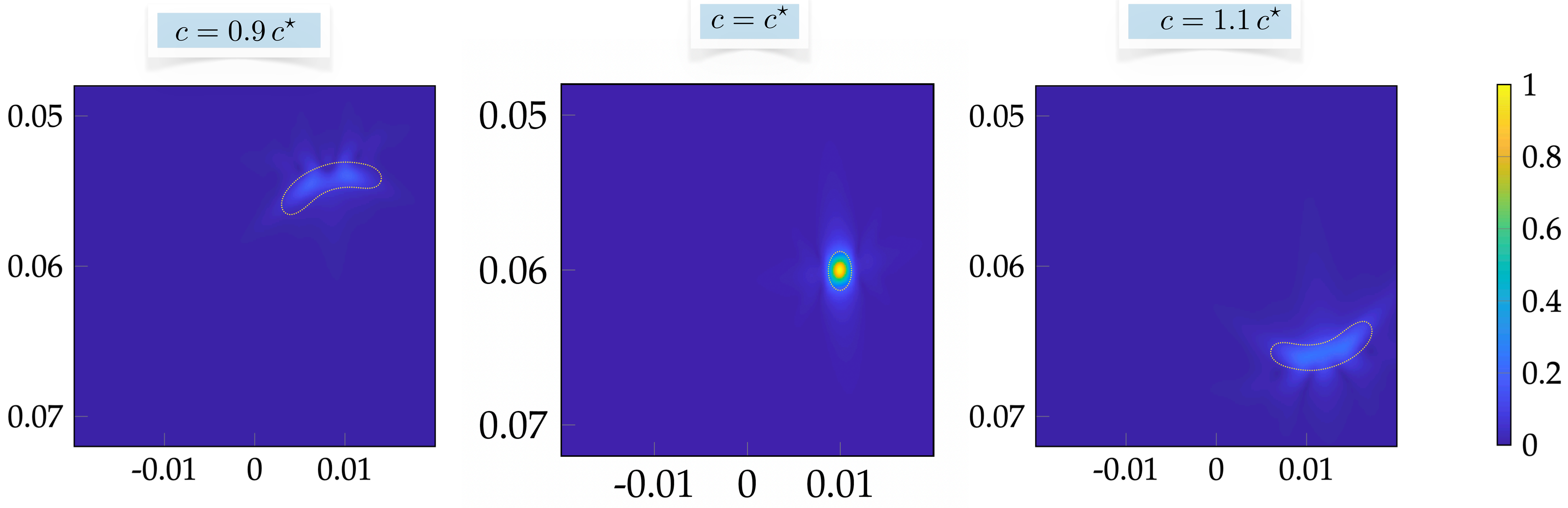


Localisation is moved

Effects of speed of sound error on the focal spot

$$\begin{aligned} D' &\longrightarrow D' \\ z &\longmapsto \left|F^{\frac{c}{c^\star}}(z, y_0)\right| \end{aligned}$$

$y_0 = (0.01, 0.06) \in D$



Localisation is moved

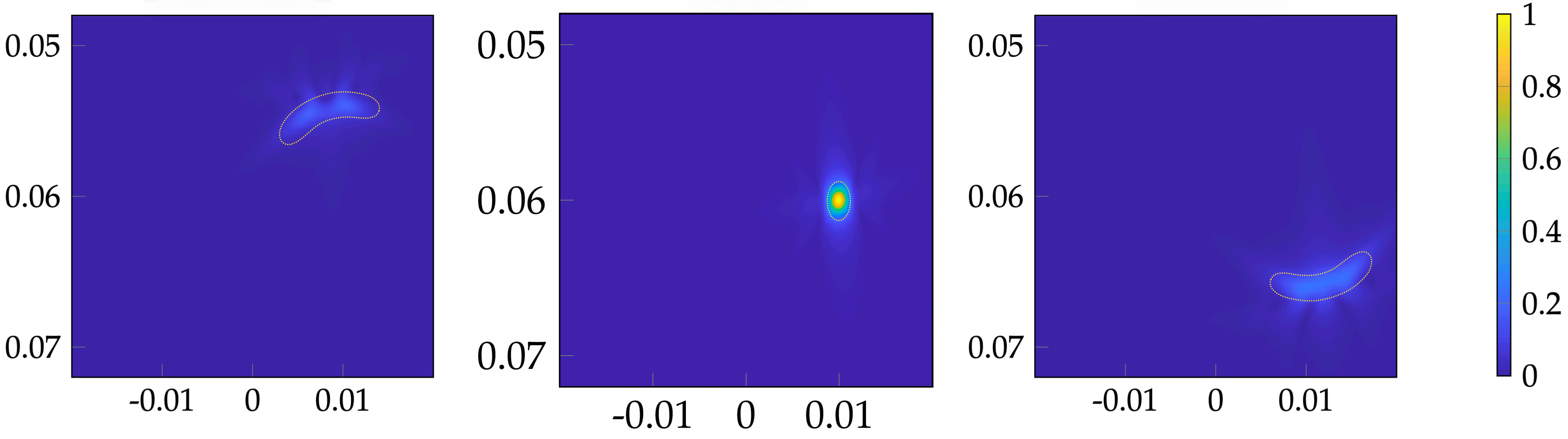
Height of peak decreased

Effects of speed of sound error on the focal spot

$$\begin{aligned} D' &\longrightarrow D' \\ z &\longmapsto \left|F^{\frac{c}{c^\star}}(z, y_0)\right| \end{aligned}$$

$y_0 = (0.01, 0.06) \in D$

$c = 0.9\, c^\star$ $\quad$ $c = c^\star$ $\quad$ $c = 1.1\, c^\star$



Localisation is moved

Height of peak decreased

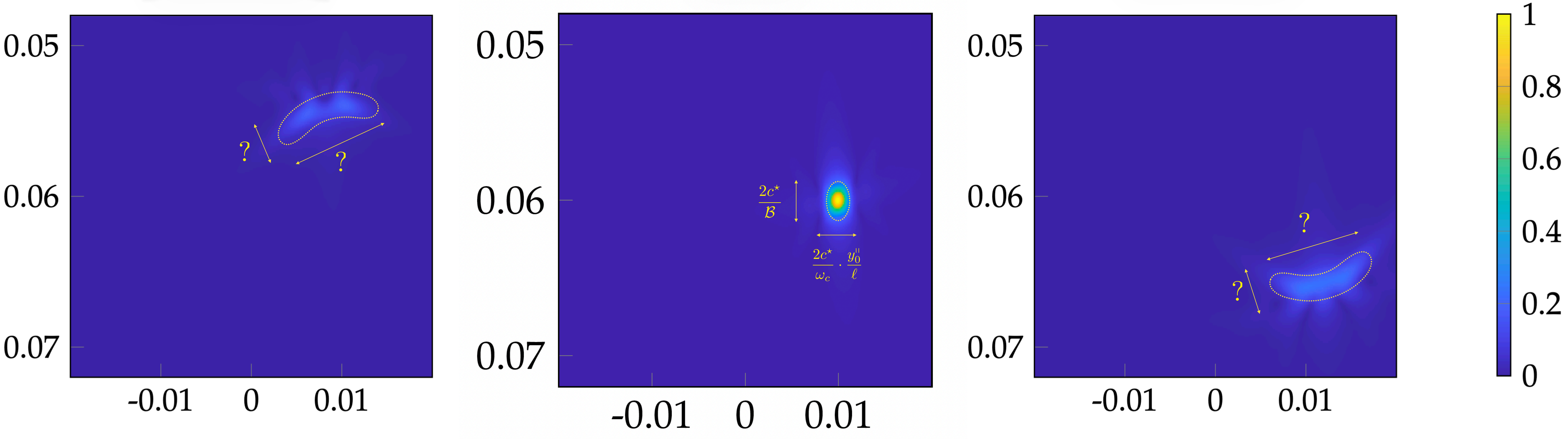
Size is increased

Effects of speed of sound error on the focal spot

$$\begin{aligned} D' &\longrightarrow D' \\ z &\longmapsto \left| F^{\frac{c}{c^\star}}(z, y_0) \right| \end{aligned}$$

$y_0 = (0.01, 0.06) \in D$

$c = 0.9\, c^\star$ | $c = c^\star$ | $c = 1.1\, c^\star$



Localisation is moved

Height of peak decreased

Size is increased

Quantification of these effects ?

The focal spot

Definition : Focal spot at threshold δ

Consider $0 < \delta \ll 1$ and $y_0 \in D$. There exists a domain $D'_\delta(y_0) \subset D'$ such that

$$\int_{D' \setminus \overline{D'_\delta(y_0)}} \left| F^{c^\star}(z, y_0) \right| \mathrm{d}z < \delta \int_{D'} \left| F^{c^\star}(z, y_0) \right| \mathrm{d}z \quad \text{and} \quad \frac{|D'_\delta(y_0)|}{|D'|} \ll \delta.$$

The focal spot

Definition : Focal spot at threshold δ

Consider $0 < \delta \ll 1$ and $y_0 \in D$. There exists a domain $D'_\delta(y_0) \subset D'$ such that

$$\int_{D' \setminus \overline{D'_\delta(y_0)}} \left| F^{c^\star}(z, y_0) \right| \mathrm{d}z < \delta \int_{D'} \left| F^{c^\star}(z, y_0) \right| \mathrm{d}z \quad \text{and} \quad \frac{|D'_\delta(y_0)|}{|D'|} \ll \delta.$$

The function $z \longmapsto |F^\star(z, y_0)|$ is of order 1 in the focal spot and « very small » outside.

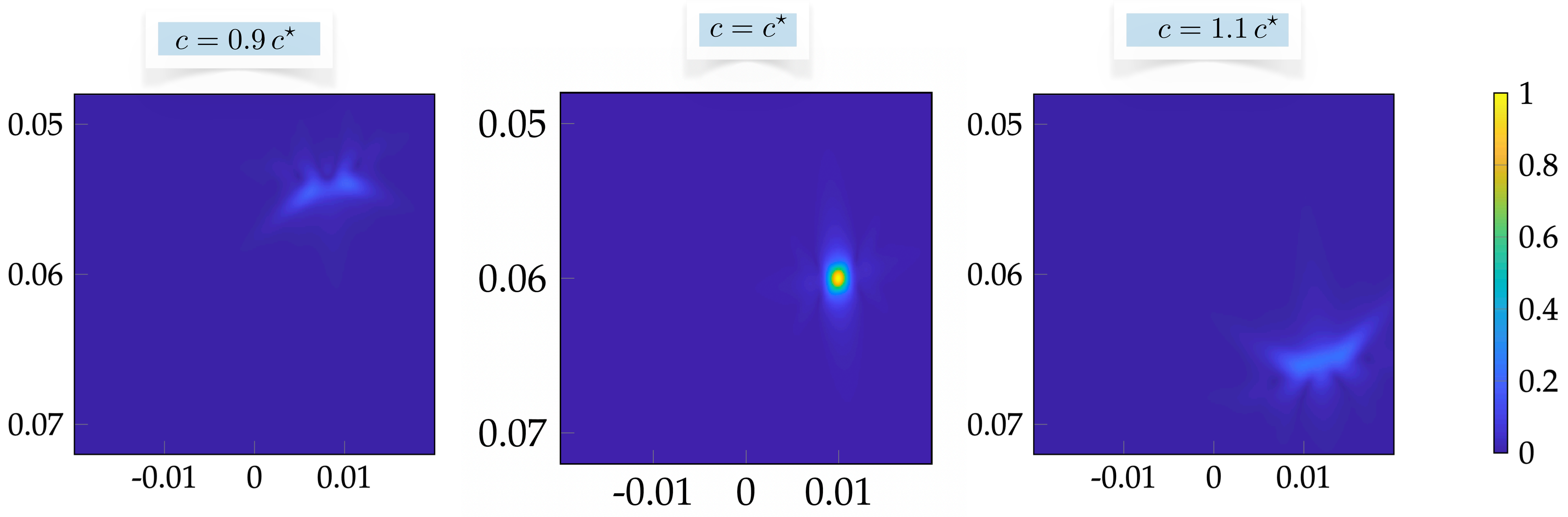
The focal spot

Definition : Focal spot at threshold δ

Consider $0 < \delta \ll 1$ and $y_0 \in D$. There exists a domain $D'_\delta(y_0) \subset D'$ such that

$$\int_{D' \setminus \overline{D'_\delta(y_0)}} \left| F^{c^\star}(z, y_0) \right| \mathrm{d}z < \delta \int_{D'} \left| F^{c^\star}(z, y_0) \right| \mathrm{d}z \quad \text{and} \quad \frac{|D'_\delta(y_0)|}{|D'|} \ll \delta.$$

The function $z \longmapsto |F^\star(z, y_0)|$ is of order 1 in the focal spot and « very small » outside.



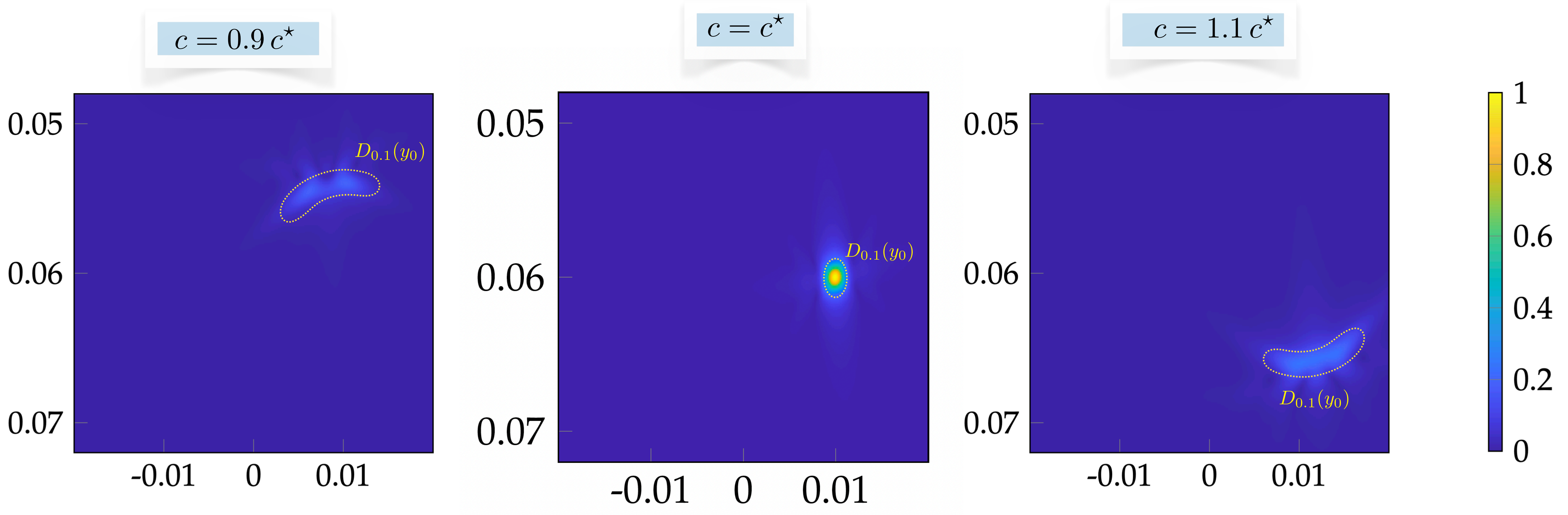
The focal spot

Definition : Focal spot at threshold δ

Consider $0 < \delta \ll 1$ and $y_0 \in D$. There exists a domain $D'_\delta(y_0) \subset D'$ such that

$$\int_{D' \setminus \overline{D'_\delta(y_0)}} \left| F^{c^\star}(z, y_0) \right| \mathrm{d}z < \delta \int_{D'} \left| F^{c^\star}(z, y_0) \right| \mathrm{d}z \quad \text{and} \quad \frac{|D'_\delta(y_0)|}{|D'|} \ll \delta.$$

The function $z \longmapsto \left| F^\star(z, y_0) \right|$ is of order 1 in the focal spot and « very small » outside.



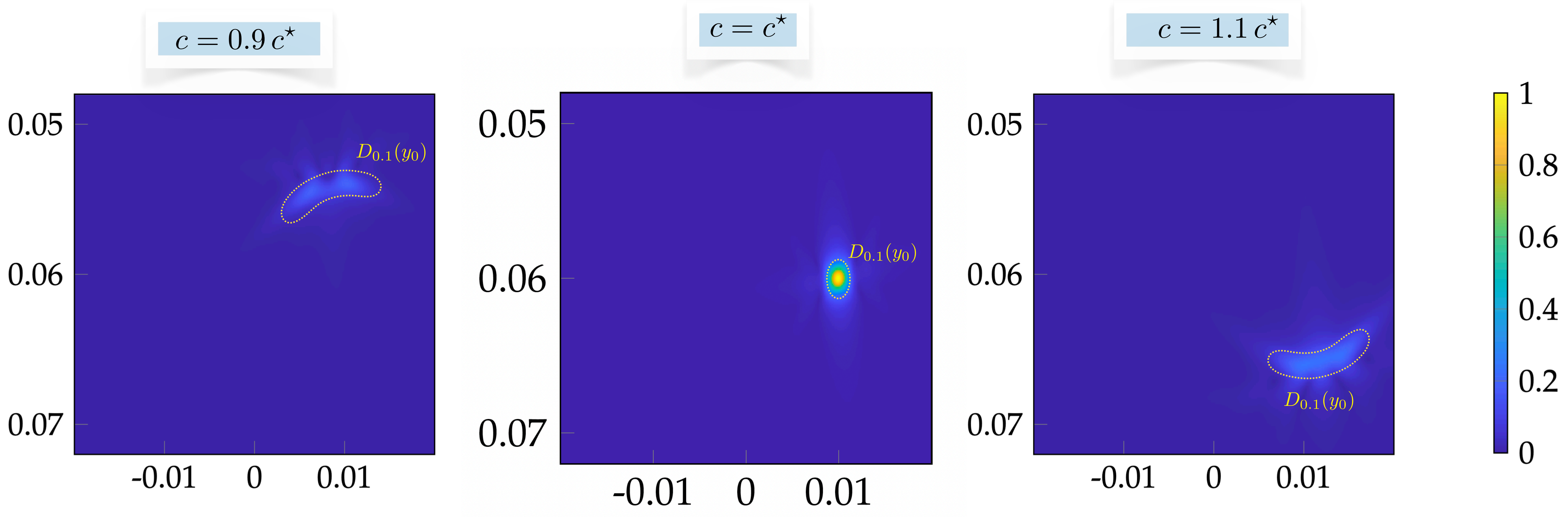
The focal spot

Definition : Focal spot at threshold δ

Consider $0 < \delta \ll 1$ and $y_0 \in D$. There exists a domain $D'_\delta(y_0) \subset D'$ such that

$$\int_{D' \setminus \overline{D'_\delta(y_0)}} \left| F^{c^\star}(z, y_0) \right| \mathrm{d}z < \delta \int_{D'} \left| F^{c^\star}(z, y_0) \right| \mathrm{d}z \quad \text{and} \quad \frac{|D'_\delta(y_0)|}{|D'|} \ll \delta.$$

The function $z \longmapsto |F^\star(z, y_0)|$ is of order 1 in the focal spot and « very small » outside.



The part of D' (the image) affected by the point $y_0 \in D$

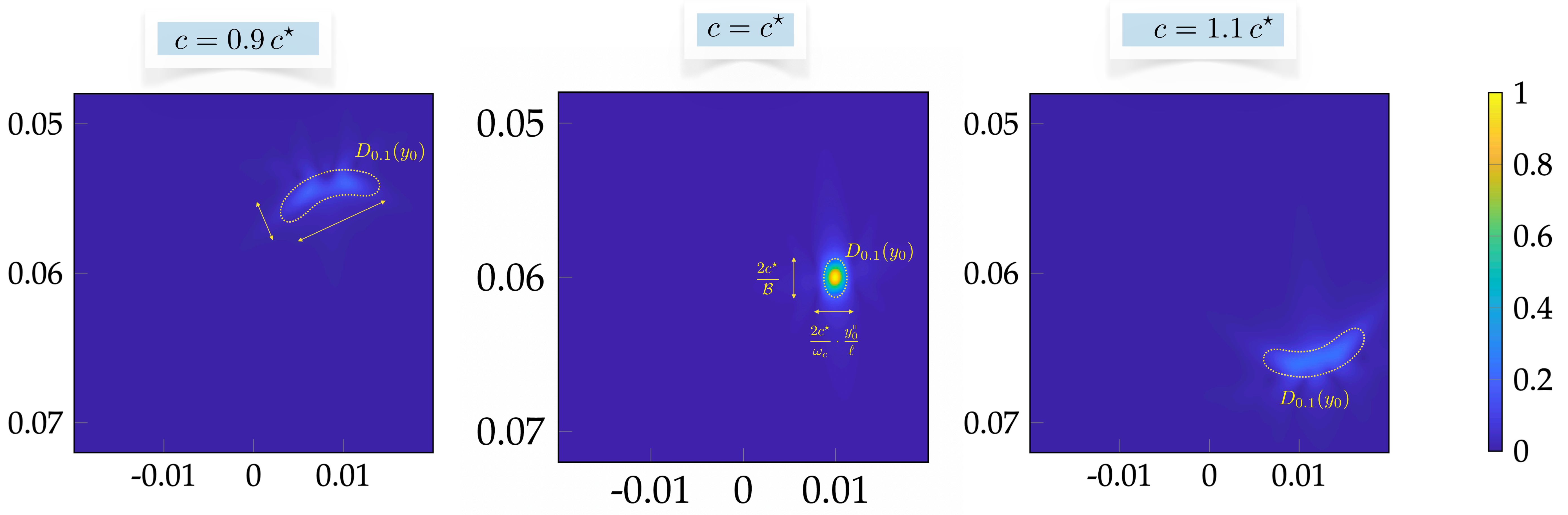
The focal spot

Definition : Focal spot at threshold δ

Consider $0 < \delta \ll 1$ and $y_0 \in D$. There exists a domain $D'_\delta(y_0) \subset D'$ such that

$$\int_{D' \setminus \overline{D'_\delta(y_0)}} \left|F^{c^\star}(z, y_0)\right| \mathrm{d}z < \delta \int_{D'} \left|F^{c^\star}(z, y_0)\right| \mathrm{d}z \quad \text{and} \quad \frac{|D'_\delta(y_0)|}{|D'|} \ll \delta.$$

The function $z \longmapsto |F^\star(z, y_0)|$ is of order 1 in the focal spot and « very small » outside.



The part of D' (the image) affected by the point $y_0 \in D$

Asymptotic analysis of the phase

Asymptotic analysis of the phase

$$F^{\frac{c}{c^\star}}(z,y) = \int_{\mathcal{B}} \omega^2 \left(\int_{\mathcal{P}} \Gamma^{\frac{\omega}{c}}(z,x_r)\overline{G^\star(y,x_r)}\mathrm{d}x_r \right)^2 \mathrm{d}\omega \qquad z,y \in D' \times D.$$

Asymptotic analysis of the phase

$$F^{\frac{c}{c^\star}}(z,y) = \int_{\mathcal{B}} \omega^2 \left(\int_{\mathcal{P}} \Gamma^{\frac{\omega}{c}}(z,x_r)\overline{G^\star(y,x_r)}\mathrm{d}x_r \right)^2 \mathrm{d}\omega \qquad z,y \in D' \times D.$$

$$\int_{\mathcal{B}} \left(\int_{\mathcal{P}} e^{i\omega\left(\frac{|z-x_r|}{c} - \frac{|y-x_r|}{c^\star}\right)}\mathrm{d}x_r \right)^2 \mathrm{d}\omega$$

Focal spot

$$\boxed{\frac{|z-x_r|}{c} - \frac{|y-x_r|}{c^\star} = \mathcal{O}\left(\frac{1}{\omega}\right)}$$

Asymptotic analysis of the phase

$$F^{\frac{c}{c^\star}}(z,y) = \int_{\mathcal{B}} \omega^2 \left(\int_{\mathcal{P}} \Gamma^{\frac{\omega}{c}}(z,x_r)\overline{G^\star(y,x_r)}\mathrm{d}x_r \right)^2 \mathrm{d}\omega \qquad z,y \in D' \times D.$$

$$\int_{\mathcal{B}} \left(\int_{\mathcal{P}} e^{i\omega\left(\frac{|z-x_r|}{c} - \frac{|y-x_r|}{c^\star}\right)}\mathrm{d}x_r \right)^2 \mathrm{d}\omega$$

Focal spot

$$\frac{|z-x_r|}{c} - \frac{|y-x_r|}{c^\star} = \mathcal{O}\left(\frac{1}{\omega}\right)$$

Paraxial asymptotic regime ($\eta \ll 1$)

Bandwidth $\mathcal{B} := \frac{\mathcal{B}_0}{\eta}$ with $\mathcal{B}_0 := [\omega_0 - \frac{B}{2}, \omega_0 + \frac{B}{2}]$

Probe $\mathcal{P} := \eta^{\frac{1}{2}}\mathcal{P}_0$ with $\mathcal{P}_0 := [-\frac{a_0}{2}, \frac{a_0}{2}]^{d-1}$

Points $y := (y_\eta^\perp, y^{''}) := (\eta^{\frac{1}{2}} y^\perp, y^{''})$

Scatterers $\varepsilon := o(\eta)$

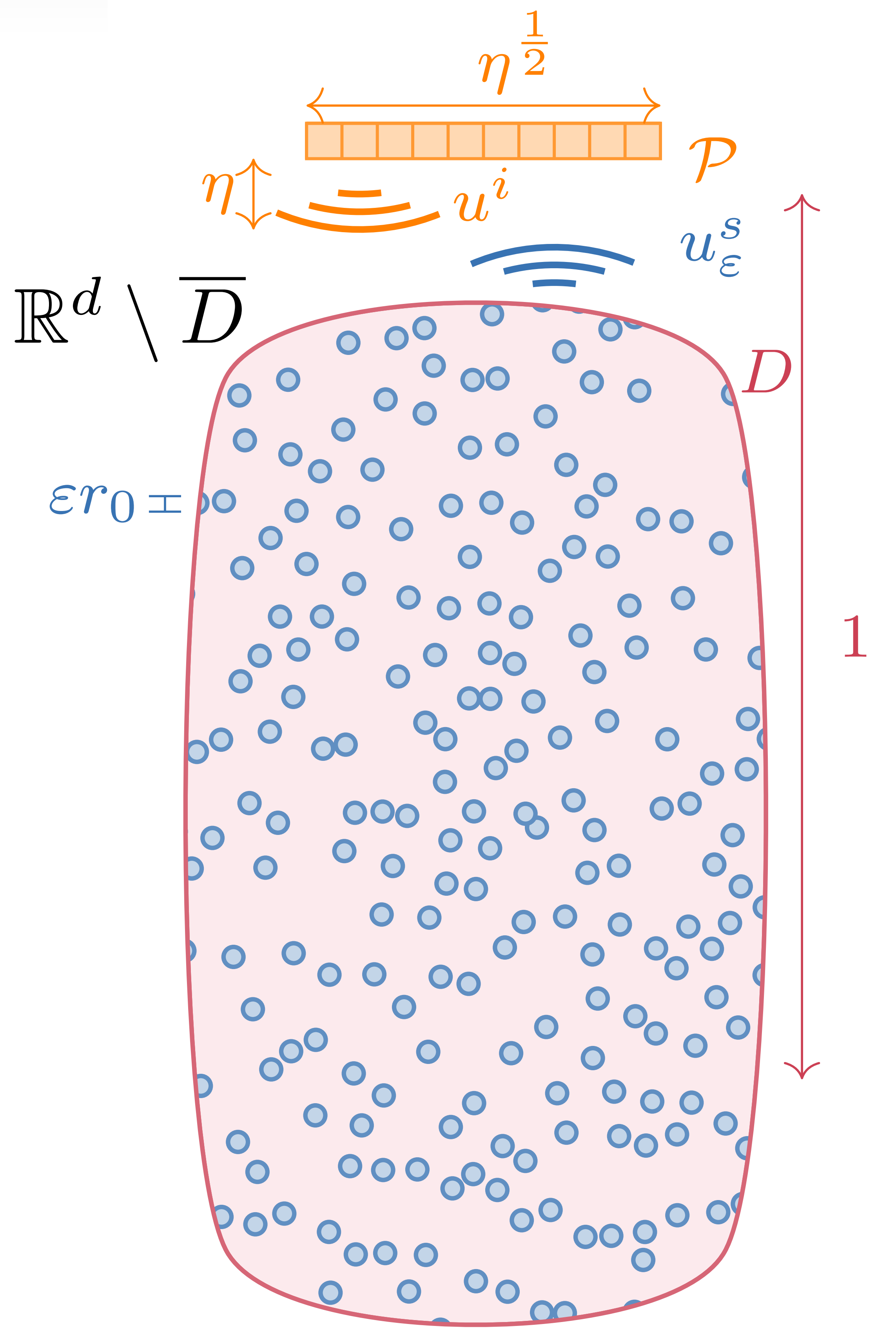
Asymptotic analysis of the phase

$$F^{\frac{c}{c^\star}}(z,y) = \int_{\mathcal{B}} \omega^2 \left(\int_{\mathcal{P}} \Gamma^{\frac{\omega}{c}}(z,x_r)\overline{G^\star(y,x_r)}\mathrm{d}x_r \right)^2 \mathrm{d}\omega \qquad z,y \in D' \times D.$$

$$\int_{\mathcal{B}} \left(\int_{\mathcal{P}} e^{i\omega\left(\frac{|z-x_r|}{c} - \frac{|y-x_r|}{c^\star}\right)}\mathrm{d}x_r \right)^2 \mathrm{d}\omega$$

Focal spot

$$\frac{|z-x_r|}{c} - \frac{|y-x_r|}{c^\star} = \mathcal{O}\left(\frac{1}{\omega}\right)$$



Paraxial asymptotic regime ($\eta \ll 1$)

Bandwidth $\mathcal{B} := \frac{\mathcal{B}_0}{\eta}$ with $\mathcal{B}_0 := [\omega_0 - \frac{B}{2}, \omega_0 + \frac{B}{2}]$

Probe $\mathcal{P} := \eta^{\frac{1}{2}}\mathcal{P}_0$ with $\mathcal{P}_0 := [-\frac{a_0}{2}, \frac{a_0}{2}]^{d-1}$

Points $y := (y^\perp_\eta, y^{||}) := (\eta^{\frac{1}{2}} y^\perp, y^{||})$

Scatterers $\varepsilon := o(\eta)$

Parameter	**Typical value**
$\frac{1}{2\pi}\mathcal{B}$	$[2,6]\mathrm{MHz}$
$c^\star$	$1500\,\mathrm{m\,s^{-1}}$
$\lvert\mathcal{P}\rvert$	$5\,\mathrm{cm}$
$y^{\lvert\lvert}$	$10\,\mathrm{cm}$
$y^\perp_\eta$	$5\,\mathrm{cm}$
εr_0	$10\,\mu\mathrm{m}$

Center of the focal spot at a given speed c

$y = (\eta^{\frac{1}{2}} y^{\perp}, y^{\parallel})$ $\quad$ $z = (\eta^{\frac{1}{2}} z^{\perp}, z^{\parallel})$

Point in the physical domain (fixed) $\quad$ Pixel on the image

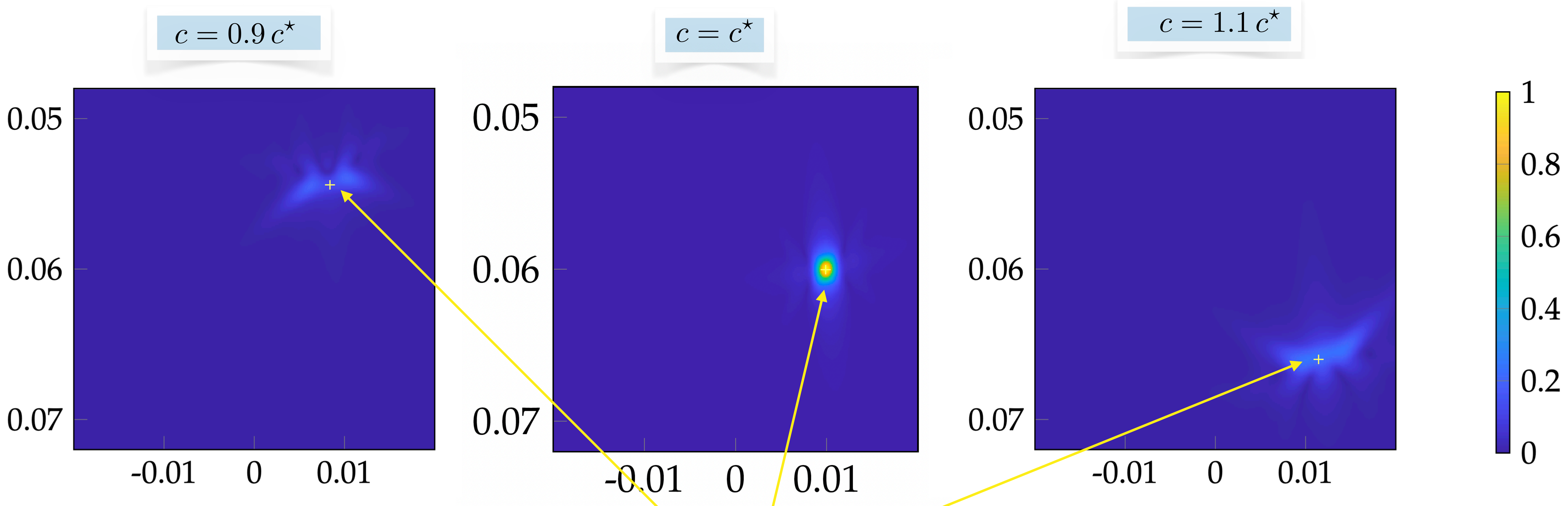
Rewriting the phase: $\Phi(z, y) = \tilde{\Phi}\left(z - \varphi_c(y)\right)$

Center of focal spot in the image

Center of the focal spot at a given speed c

$y = (\eta^{\frac{1}{2}} y^{\perp}, y^{\shortparallel})$ $\quad z = (\eta^{\frac{1}{2}} z^{\perp}, z^{\shortparallel})$ $\qquad$ Rewriting the phase: $\Phi(z, y) = \tilde{\Phi}\left(z - \varphi_c(y)\right)$

Point in the physical domain (fixed) $\quad$ Pixel on the image $\quad$ Center of focal spot in the image



$\varphi_c(y)$

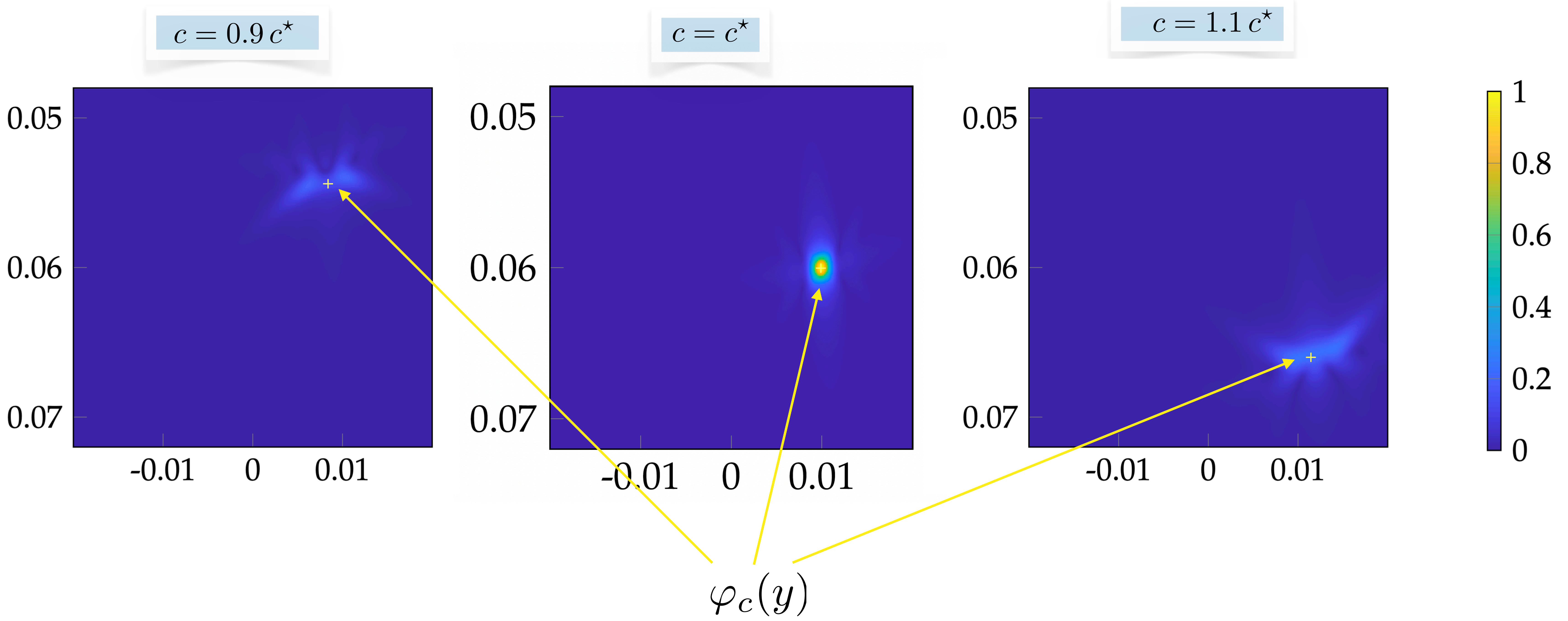
Center of the focal spot at a given speed c

$y = (\eta^{\frac{1}{2}} y^{\perp}, y^{\parallel})$ $\quad z = (\eta^{\frac{1}{2}} z^{\perp}, z^{\parallel})$

Point in the physical domain (fixed) Pixel on the image

Rewriting the phase: $\Phi(z,y) = \tilde{\Phi}\left(z - \varphi_c(y)\right)$

Center of focal spot in the image



Asymptotic expansion of travel times in the paraxial regime

$$\frac{|z - x_r|}{c} - \frac{|y - x_r|}{c^{\star}} = \sum_k \eta^k f_k(z - \varphi_c(y))$$

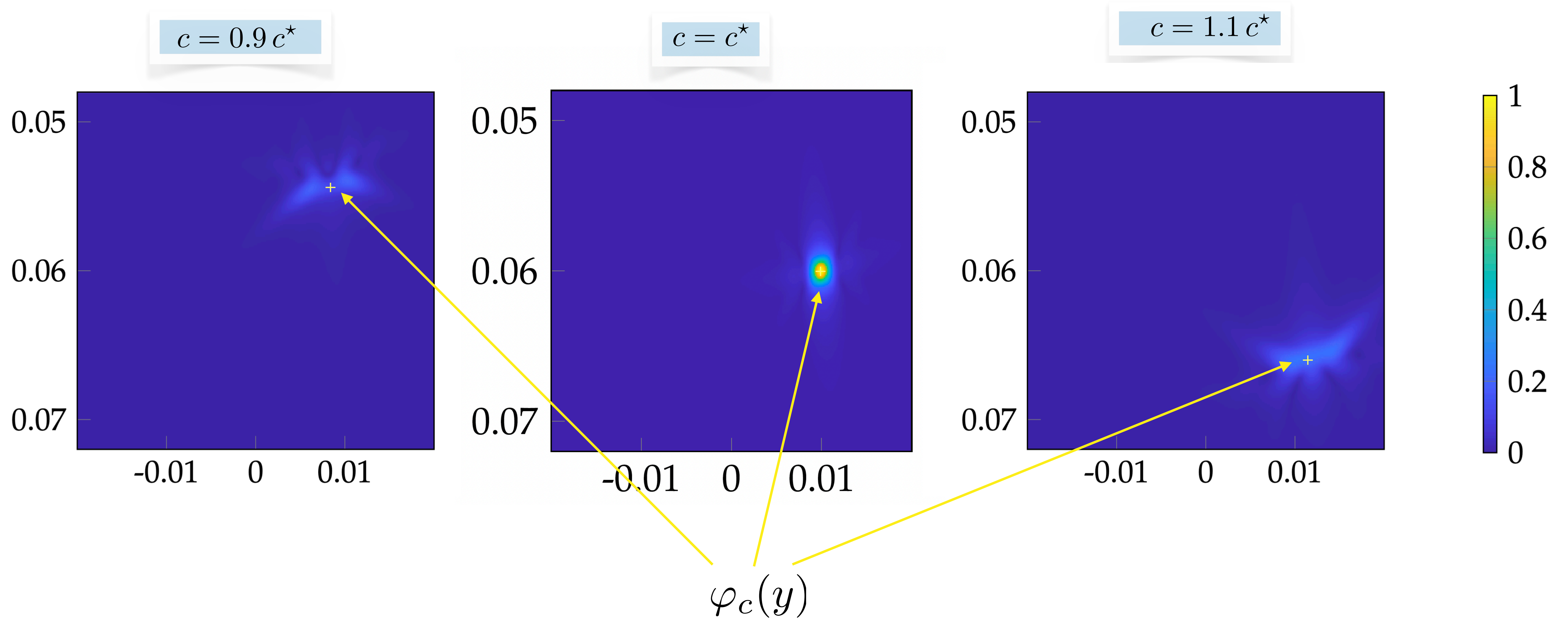
Center of the focal spot at a given speed c

$y = (\eta^{\frac{1}{2}} y^{\perp}, y^{\parallel})$ $\quad z = (\eta^{\frac{1}{2}} z^{\perp}, z^{\parallel})$

Rewriting the phase: $\Phi(z,y) = \tilde{\Phi}\,(z - \varphi_c(y))$

Point in the physical domain (fixed) Pixel on the image

Center of focal spot in the image



Asymptotic expansion of travel times in the paraxial regime

$$\frac{|z - x_r|}{c} - \frac{|y - x_r|}{c^{\star}} = \sum_k \eta^k f_k(z - \varphi_c(y))$$

Center of the focal spot

$$\varphi_c(y) = \left(\left(\frac{c}{c^{\star}} \right)^2 \eta^{\frac{1}{2}} y^{\perp}, \frac{c}{c^{\star}} y \right)$$

Expansion of the PSF around the center of the focal spot

Analytic approximation of the PSF

$$F^c(z,y) = \eta^{-1}\left(\frac{c}{c^\star}\right)^2 \frac{\ell_0^4}{(16\pi^2|z||\varphi_c(y)|)^2}\int_{\mathcal{B}_0}\omega^2 e^{i\frac{2\omega}{\eta c}(|z|-|\varphi_c(y)|)} e^{i\frac{\omega}{c}\left(1-\left(\frac{c^\star}{c}\right)^2\right)\frac{|\varphi_c(y)^\perp|^2}{|\varphi_c(y)|}}$$

$$\mathcal{G}^2\left(\frac{\omega\ell_0}{c}\left(\frac{z^\perp}{|z|}-\frac{\varphi_c(y)^\perp}{|\varphi_c(y)|}\right),\frac{\omega\ell_0^2}{c}\left(\frac{1}{|z|}-\left(\frac{c}{c^\star}\right)^2\frac{1}{|\varphi_c(y)|}\right)\right)\mathrm{d}\omega + \mathcal{O}(1).$$

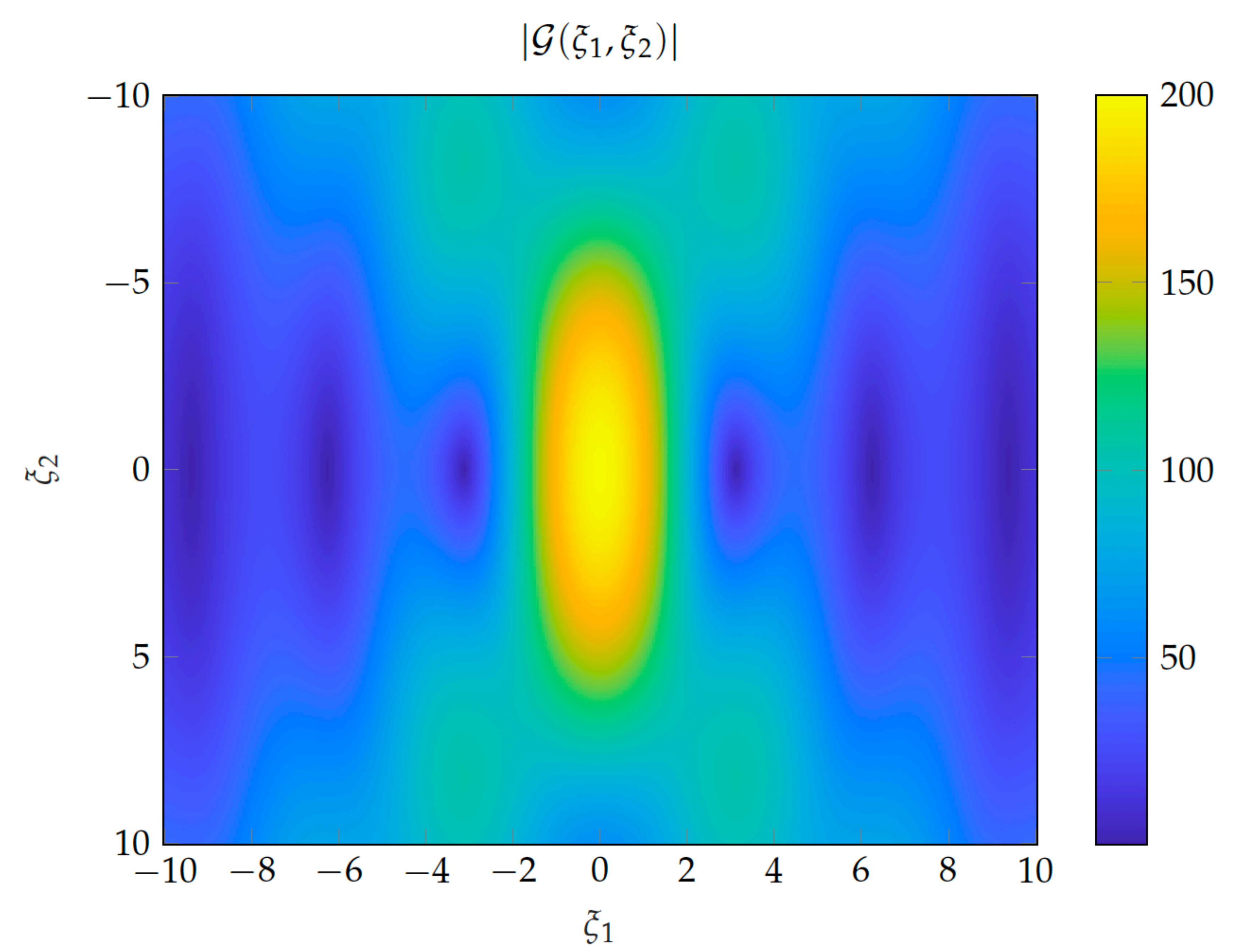
$$\mathcal{G}(\xi_1,\xi_2) := \int_{[-1,1]^2} e^{-ix_e^\perp\cdot\xi_1 + i\frac{\left|x_e^\perp\right|^2}{2}\xi_2}\mathrm{d}x_e^\perp$$

$$\xi_1\in\mathbb{R}^{d-1}, \xi_2\in\mathbb{R}.$$

Expansion of the PSF around the center of the focal spot

Analytic approximation of the PSF

$$F^c(z,y) = \eta^{-1}\left(\frac{c}{c^\star}\right)^2 \frac{\ell_0^4}{\left(16\pi^2|z||\varphi_c(y)|\right)^2}\int_{\mathcal{B}_0}\omega^2 e^{i\frac{2\omega}{\eta c}(|z|-|\varphi_c(y)|)} e^{i\frac{\omega}{c}\left(1-\left(\frac{c^\star}{c}\right)^2\right)\frac{|\varphi_c(y)^\perp|^2}{|\varphi_c(y)|}}$$
$$\mathcal{G}^2\left(\frac{\omega\ell_0}{c}\left(\frac{z^\perp}{|z|}-\frac{\varphi_c(y)^\perp}{|\varphi_c(y)|}\right), \frac{\omega\ell_0^2}{c}\left(\frac{1}{|z|}-\left(\frac{c}{c^\star}\right)^2\frac{1}{|\varphi_c(y)|}\right)\right)\mathrm{d}\omega + \mathcal{O}(1).$$



$$\mathcal{G}(\xi_1,\xi_2) := \int_{[-1,1]^2} e^{-ix_e^\perp\cdot\xi_1 + i\frac{\left|x_e^\perp\right|^2}{2}\xi_2}\mathrm{d}x_e^\perp$$

$$\xi_1 \in \mathbb{R}^{d-1}, \xi_2 \in \mathbb{R}.$$

Expansion of the PSF around the center of the focal spot

Analytic approximation of the PSF

$$F^c(z,y) = \eta^{-1}\left(\frac{c}{c^\star}\right)^2 \frac{\ell_0^4}{(16\pi^2|z||\varphi_c(y)|)^2}\int_{\mathcal{B}_0}\omega^2 e^{i\frac{2\omega}{\eta c}(|z|-|\varphi_c(y)|)} e^{i\frac{\omega}{c}\left(1-\left(\frac{c^\star}{c}\right)^2\right)\frac{|\varphi_c(y)^\perp|^2}{|\varphi_c(y)|}}$$

$$\mathcal{G}^2\left(\frac{\omega\ell_0}{c}\left(\frac{z^\perp}{|z|}-\frac{\varphi_c(y)^\perp}{|\varphi_c(y)|}\right), \frac{\omega\ell_0^2}{c}\left(\frac{1}{|z|}-\left(\frac{c}{c^\star}\right)^2\frac{1}{|\varphi_c(y)|}\right)\right)\mathrm{d}\omega + \mathcal{O}(1).$$

$$\mathcal{G}(\xi_1,\xi_2) := \int_{[-1,1]^2} e^{-ix_e^\perp\cdot\xi_1 + i\frac{\left|x_e^\perp\right|^2}{2}\xi_2}\mathrm{d}x_e^\perp$$

$$\xi_1\in\mathbb{R}^{d-1}, \xi_2\in\mathbb{R}.$$

Expansion of the PSF around the center of the focal spot

Analytic approximation of the PSF

$$F^c(z,y) = \eta^{-1}\left(\frac{c}{c^\star}\right)^2 \frac{\ell_0^4}{(16\pi^2|z||\varphi_c(y)|)^2}\int_{\mathcal{B}_0} \omega^2 e^{i\frac{2\omega}{\eta c}(|z|-|\varphi_c(y)|)} e^{i\frac{\omega}{c}\left(1-\left(\frac{c^\star}{c}\right)^2\right)\frac{|\varphi_c(y)^\perp|^2}{|\varphi_c(y)|}}$$

$$\mathcal{G}^2\left(\frac{\omega\ell_0}{c}\left(\frac{z^\perp}{|z|}-\frac{\varphi_c(y)^\perp}{|\varphi_c(y)|}\right), \frac{\omega\ell_0^2}{c}\left(\frac{1}{|z|}-\left(\frac{c}{c^\star}\right)^2\frac{1}{|\varphi_c(y)|}\right)\right)\mathrm{d}\omega + \mathcal{O}(1).$$

$$\mathcal{G}(\xi_1,\xi_2) := \int_{[-1,1]^2} e^{-ix_e^\perp\cdot\xi_1 + i\frac{\left|x_e^\perp\right|^2}{2}\xi_2}\mathrm{d}x_e^\perp$$

$$\xi_1 \in \mathbb{R}^{d-1}, \xi_2 \in \mathbb{R}.$$

Nice expression at the center $z = \varphi_c(y)$

Expansion of the PSF around the center of the focal spot

Analytic approximation of the PSF

$$F^c(z,y) = \eta^{-1}\left(\frac{c}{c^\star}\right)^2 \frac{\ell_0^4}{(16\pi^2|z||\varphi_c(y)|)^2}\int_{\mathcal{B}_0}\omega^2 e^{i\frac{2\omega}{\eta c}(|z|-|\varphi_c(y)|)}e^{i\frac{\omega}{c}\left(1-\left(\frac{c^\star}{c}\right)^2\right)\frac{|\varphi_c(y)^\perp|^2}{|\varphi_c(y)|}}$$
$$\mathcal{G}^2\left(\frac{\omega\ell_0}{c}\left(\frac{z^\perp}{|z|}-\frac{\varphi_c(y)^\perp}{|\varphi_c(y)|}\right),\frac{\omega\ell_0^2}{c}\left(\frac{1}{|z|}-\left(\frac{c}{c^\star}\right)^2\frac{1}{|\varphi_c(y)|}\right)\right)\mathrm{d}\omega+\mathcal{O}(1).$$

$$\mathcal{G}(\xi_1,\xi_2) := \int_{[-1,1]^2} e^{-ix_e^\perp\cdot\xi_1+i\frac{|x_e^\perp|^2}{2}\xi_2}\mathrm{d}x_e^\perp$$
$$\xi_1\in\mathbb{R}^{d-1},\xi_2\in\mathbb{R}.$$

Nice expression at the center $z=\varphi_c(y)$

Size and shape of the focal spot $z=\varphi_c(z)+\left(\eta^{\frac{1}{2}}\Delta z^\perp,\eta\Delta z^{\shortparallel}\right)$

Expansion of the PSF around the center of the focal spot

Analytic approximation of the PSF

$$F^c(z,y) = \eta^{-1}\left(\frac{c}{c^\star}\right)^2 \frac{\ell_0^4}{(16\pi^2|z||\varphi_c(y)|)^2}\int_{\mathcal{B}_0} \omega^2 e^{i\frac{2\omega}{\eta c}(|z|-|\varphi_c(y)|)} e^{i\frac{\omega}{c}\left(1-\left(\frac{c^\star}{c}\right)^2\right)\frac{|\varphi_c(y)^\perp|^2}{|\varphi_c(y)|}}$$
$$\mathcal{G}^2\left(\frac{\omega\ell_0}{c}\left(\frac{z^\perp}{|z|}-\frac{\varphi_c(y)^\perp}{|\varphi_c(y)|}\right), \frac{\omega\ell_0^2}{c}\left(\frac{1}{|z|}-\left(\frac{c}{c^\star}\right)^2\frac{1}{|\varphi_c(y)|}\right)\right)\mathrm{d}\omega + \mathcal{O}(1).$$

$$\mathcal{G}(\xi_1,\xi_2) := \int_{[-1,1]^2} e^{-ix_e^\perp\cdot\xi_1 + i\frac{|x_e^\perp|^2}{2}\xi_2}\mathrm{d}x_e^\perp$$
$$\xi_1 \in \mathbb{R}^{d-1}, \xi_2 \in \mathbb{R}.$$

Nice expression at the center $z = \varphi_c(y)$

Size and shape of the focal spot $z = \varphi_c(z) + \left(\eta^{\frac{1}{2}}\Delta z^\perp, \eta\Delta z^{\shortparallel}\right)$

$$F^c(z,y_0) = \eta^{-1}\left(\frac{c^\star}{c}\right)^2 \frac{\ell_0^4}{(16\pi^2|y_0|^2)^2}\int_{\mathcal{B}_0} \omega^2 e^{i\frac{\omega}{|y_0|c^\star}|y_0^\perp|^2\left(\left(\frac{c}{c^\star}\right)^2-1\right)} e^{i\frac{2\omega}{c}\Delta z^{\shortparallel}} e^{i\frac{\omega}{|y_0|c^\star}\left(\frac{c^\star}{c}\right)^2|\Delta z^\perp|^2}$$
$$\mathcal{G}^2\left(\frac{\omega\ell_0}{c^\star|y_0|}\left(\frac{c^\star}{c}\right)^2\Delta z^\perp, \frac{\omega\ell_0^2}{c^\star|y_0|}\left(\left(\frac{c^\star}{c}\right)^2-1\right)\right)\mathrm{d}\omega + \mathcal{O}(1).$$

Expansion of the PSF around the center of the focal spot

Analytic approximation of the PSF

$$F^c(z,y) = \eta^{-1}\left(\frac{c}{c^\star}\right)^2 \frac{\ell_0^4}{(16\pi^2|z||\varphi_c(y)|)^2}\int_{\mathcal{B}_0}\omega^2 e^{i\frac{2\omega}{\eta c}(|z|-|\varphi_c(y)|)} e^{i\frac{\omega}{c}\left(1-\left(\frac{c^\star}{c}\right)^2\right)\frac{|\varphi_c(y)^\perp|^2}{|\varphi_c(y)|}}$$
$$\mathcal{G}^2\left(\frac{\omega\ell_0}{c}\left(\frac{z^\perp}{|z|}-\frac{\varphi_c(y)^\perp}{|\varphi_c(y)|}\right), \frac{\omega\ell_0^2}{c}\left(\frac{1}{|z|}-\left(\frac{c}{c^\star}\right)^2\frac{1}{|\varphi_c(y)|}\right)\right)\mathrm{d}\omega + \mathcal{O}(1).$$

$$\mathcal{G}(\xi_1,\xi_2) := \int_{[-1,1]^2} e^{-ix_e^\perp\cdot\xi_1 + i\frac{\left|x_e^\perp\right|^2}{2}\xi_2}\mathrm{d}x_e^\perp$$
$$\xi_1 \in \mathbb{R}^{d-1}, \xi_2 \in \mathbb{R}.$$

Nice expression at the center $z = \varphi_c(y)$

Size and shape of the focal spot $z = \varphi_c(z) + \left(\eta^{\frac{1}{2}}\Delta z^\perp, \eta\Delta z^{\shortparallel}\right)$

$$F^c(z,y_0) = \eta^{-1}\left(\frac{c^\star}{c}\right)^2\frac{\ell_0^4}{(16\pi^2|y_0|^2)^2}\int_{\mathcal{B}_0}\omega^2 e^{i\frac{\omega}{|y_0|c^\star}|y_0^\perp|^2\left(\left(\frac{c}{c^\star}\right)^2-1\right)} e^{i\frac{2\omega}{c}\Delta z^{\shortparallel}} e^{i\frac{\omega}{|y_0|c^\star}\left(\frac{c^\star}{c}\right)^2|\Delta z^\perp|^2}$$
$$\mathcal{G}^2\left(\frac{\omega\ell_0}{c^\star|y_0|}\left(\frac{c^\star}{c}\right)^2\Delta z^\perp, \frac{\omega\ell_0^2}{c^\star|y_0|}\left(\left(\frac{c^\star}{c}\right)^2-1\right)\right)\mathrm{d}\omega + \mathcal{O}(1).$$

Expansion of the PSF around the center of the focal spot

Analytic approximation of the PSF

$$F^c(z,y) = \eta^{-1}\left(\frac{c}{c^\star}\right)^2 \frac{\ell_0^4}{(16\pi^2|z||\varphi_c(y)|)^2}\int_{\mathcal{B}_0} \omega^2 e^{i\frac{2\omega}{\eta c}(|z|-|\varphi_c(y)|)} e^{i\frac{\omega}{c}\left(1-\left(\frac{c^\star}{c}\right)^2\right)\frac{|\varphi_c(y)^\perp|^2}{|\varphi_c(y)|}}$$
$$\mathcal{G}^2\left(\frac{\omega\ell_0}{c}\left(\frac{z^\perp}{|z|}-\frac{\varphi_c(y)^\perp}{|\varphi_c(y)|}\right), \frac{\omega\ell_0^2}{c}\left(\frac{1}{|z|}-\left(\frac{c}{c^\star}\right)^2\frac{1}{|\varphi_c(y)|}\right)\right)\mathrm{d}\omega + \mathcal{O}(1).$$

$$\mathcal{G}(\xi_1,\xi_2) := \int_{[-1,1]^2} e^{-ix_e^\perp\cdot\xi_1 + i\frac{|x_e^\perp|^2}{2}\xi_2}\mathrm{d}x_e^\perp$$
$$\xi_1 \in \mathbb{R}^{d-1}, \xi_2 \in \mathbb{R}.$$

Nice expression at the center $z = \varphi_c(y)$ ← Useful now

Size and shape of the focal spot $z = \varphi_c(z) + \left(\eta^{\frac{1}{2}}\Delta z^\perp, \eta\Delta z^{\shortparallel}\right)$ ← Useful later

$$F^c(z,y_0) = \eta^{-1}\left(\frac{c^\star}{c}\right)^2 \frac{\ell_0^4}{(16\pi^2|y_0|^2)^2}\int_{\mathcal{B}_0} \omega^2 e^{i\frac{\omega}{|y_0|c^\star}|y_0^\perp|^2\left(\left(\frac{c}{c^\star}\right)^2-1\right)} e^{i\frac{2\omega}{c}\Delta z^{\shortparallel}} e^{i\frac{\omega}{|y_0|c^\star}\left(\frac{c^\star}{c}\right)^2|\Delta z^\perp|^2}$$
$$\mathcal{G}^2\left(\frac{\omega\ell_0}{c^\star|y_0|}\left(\frac{c^\star}{c}\right)^2\Delta z^\perp, \frac{\omega\ell_0^2}{c^\star|y_0|}\left(\left(\frac{c^\star}{c}\right)^2-1\right)\right)\mathrm{d}\omega + \mathcal{O}(1).$$

PSF amplitude estimator

Definition : Confocal $\mathscr{F}$ trace at y_0

$$\begin{aligned} [c_{\min}, c_{\max}] &\longrightarrow \mathbb{C} \\ c &\longmapsto \mathcal{F}(c) := F^{\frac{c}{c^\star}}\left(\phi_c(y_0), y_0\right) \end{aligned}$$

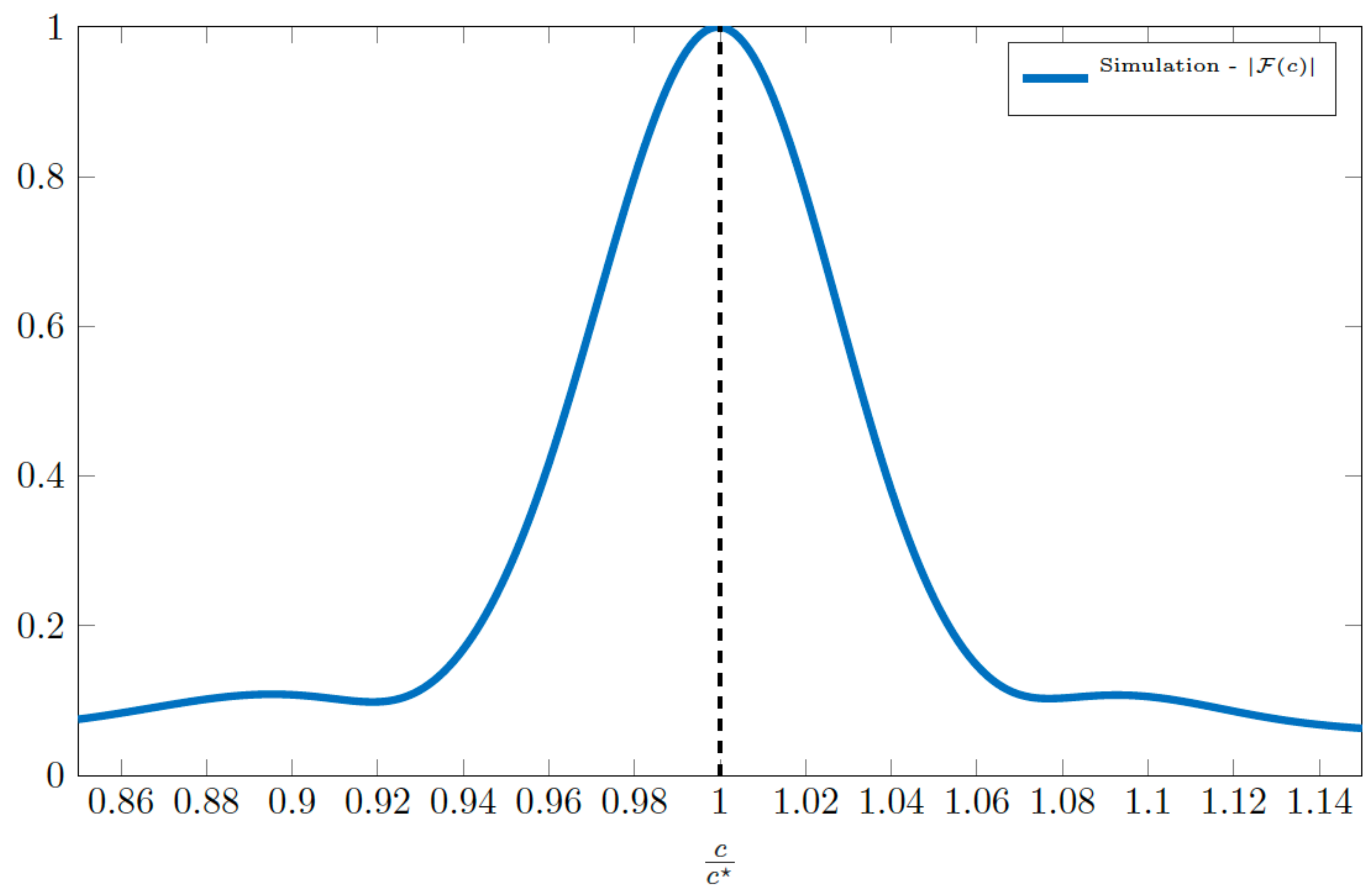
with $y_0 \in D$

PSF amplitude estimator

Definition : Confocal $\mathscr{F}$ trace at y_0

$$\begin{aligned}[c_{\min}, c_{\max}] &\longrightarrow \mathbb{C}\\ c &\longmapsto \mathcal{F}(c) := F^{\frac{c}{c^\star}}\left(\phi_c(y_0), y_0\right)\end{aligned}$$

with $y_0 \in D$

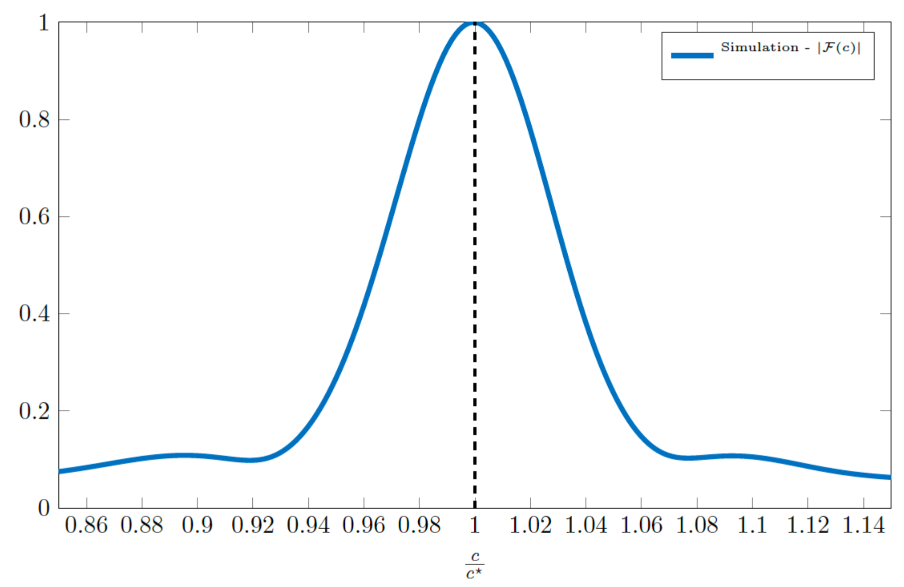


PSF amplitude estimator

Definition : Confocal $\mathscr{F}$ trace at y_0

$$\begin{aligned} [c_{\min}, c_{\max}] &\longrightarrow \mathbb{C} \\ c &\longmapsto \mathcal{F}(c) := F^{\frac{c}{c^\star}}\left(\phi_c(y_0), y_0\right) \end{aligned}$$

with $y_0 \in D$



Proposition : Amplitude Estimator

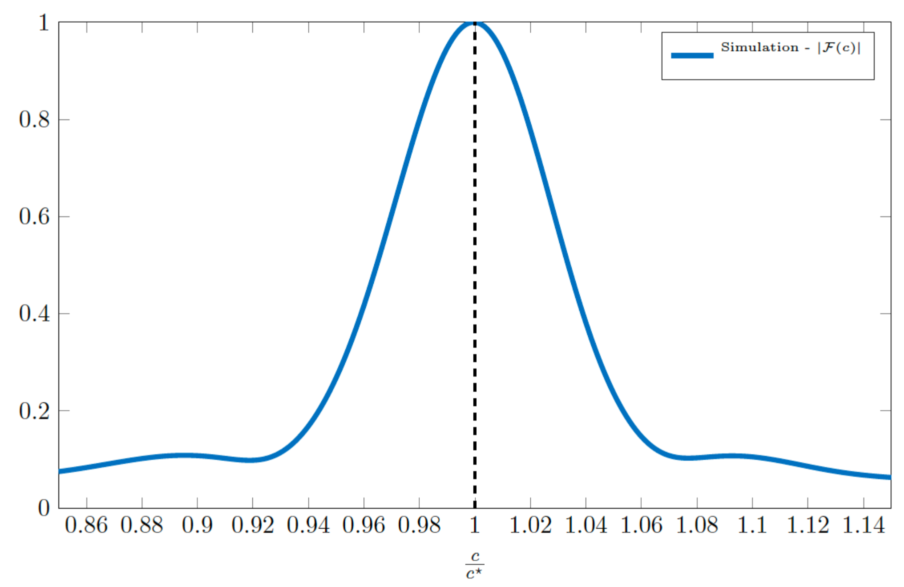
$$c^\star = \operatorname{argmax}_c |\mathcal{F}(c)| \,.$$

PSF amplitude estimator

Definition : Confocal $\mathscr{F}$ trace at y_0

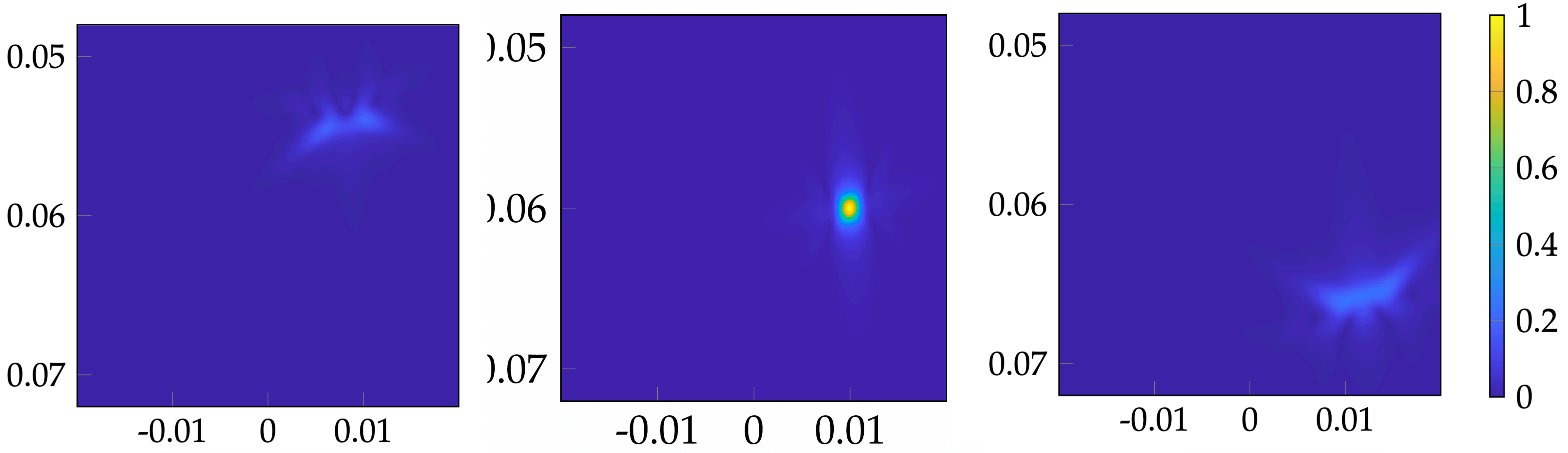
$$[c_{\min}, c_{\max}] \longrightarrow \mathbb{C}$$
$$c \longmapsto \mathcal{F}(c) := F^{\frac{c}{c^\star}}\left(\phi_c(y_0), y_0\right)$$

with $y_0 \in D$



Proposition : Amplitude Estimator

$$c^\star = \operatorname{argmax}_c |\mathcal{F}(c)| \,.$$



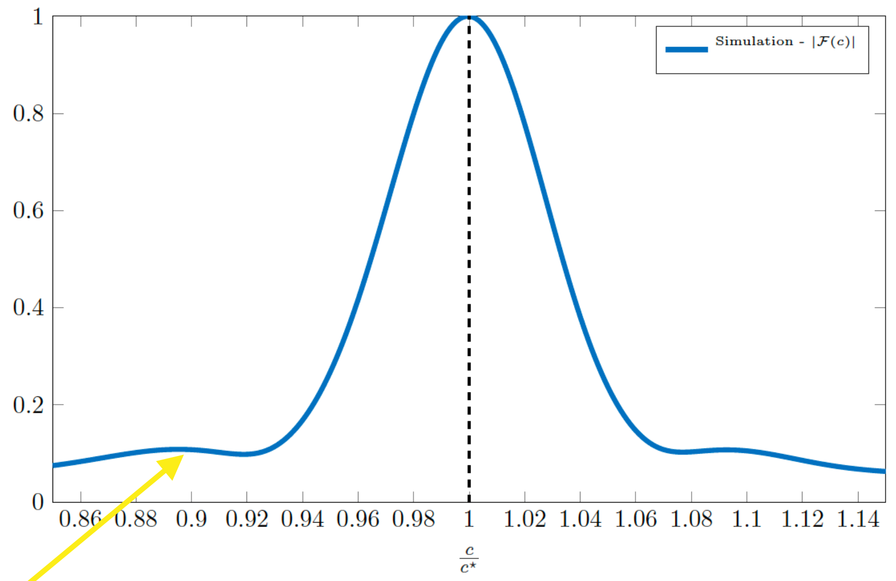
$c_1 = 0.9\,c^\star$ $c_2 = c^\star$ $c_3 = 1.1c^\star$

PSF amplitude estimator

Definition : Confocal $\mathscr{F}$ trace at y_0

$$\begin{aligned} [c_{\min}, c_{\max}] &\longrightarrow \mathbb{C} \\ c &\longmapsto \mathcal{F}(c) := F^{\frac{c}{c^\star}}\left(\phi_c(y_0), y_0\right) \end{aligned}$$

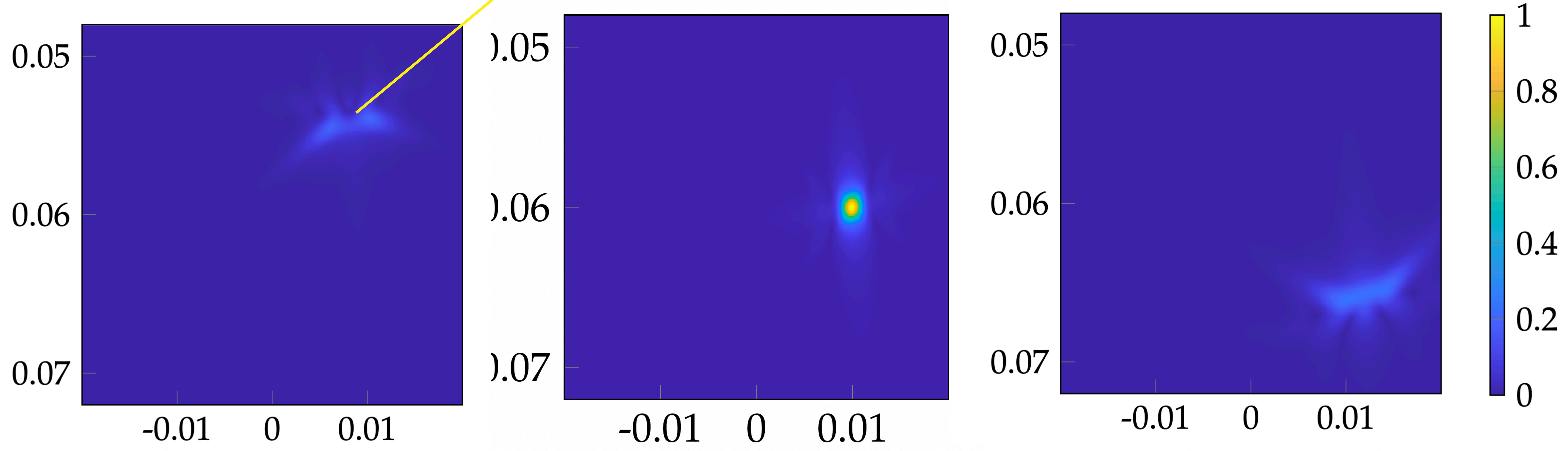
with $y_0 \in D$



Proposition : Amplitude Estimator

$$c^\star = \operatorname{argmax}_c |\mathcal{F}(c)| \,.$$

$\mathcal{F}(c_1)$



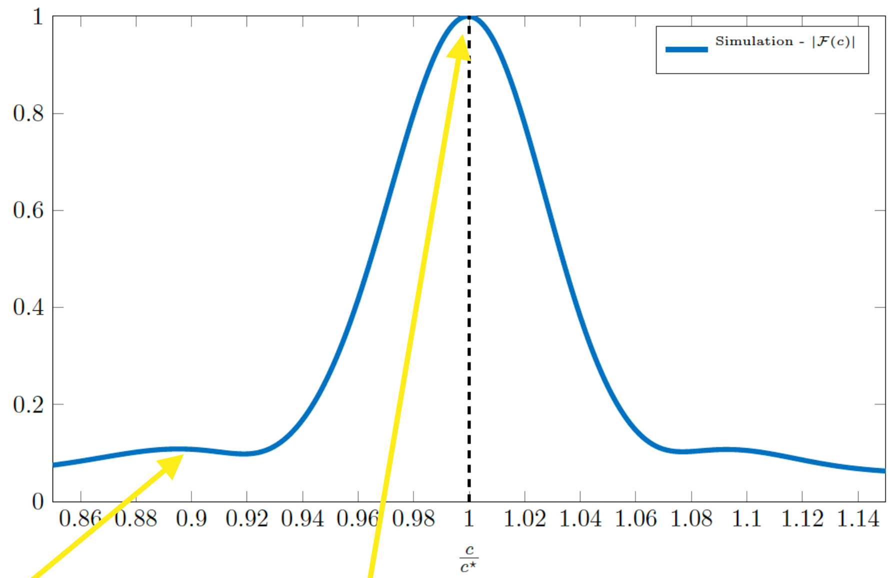
$c_1 = 0.9\, c^\star$ $\quad$ $c_2 = c^\star$ $\quad$ $c_3 = 1.1 c^\star$

PSF amplitude estimator

Definition : Confocal $\mathscr{F}$ trace at y_0

$$[c_{\min}, c_{\max}] \longrightarrow \mathbb{C}$$
$$c \longmapsto \mathcal{F}(c) := F^{\frac{c}{c^\star}}\left(\phi_c(y_0), y_0\right)$$

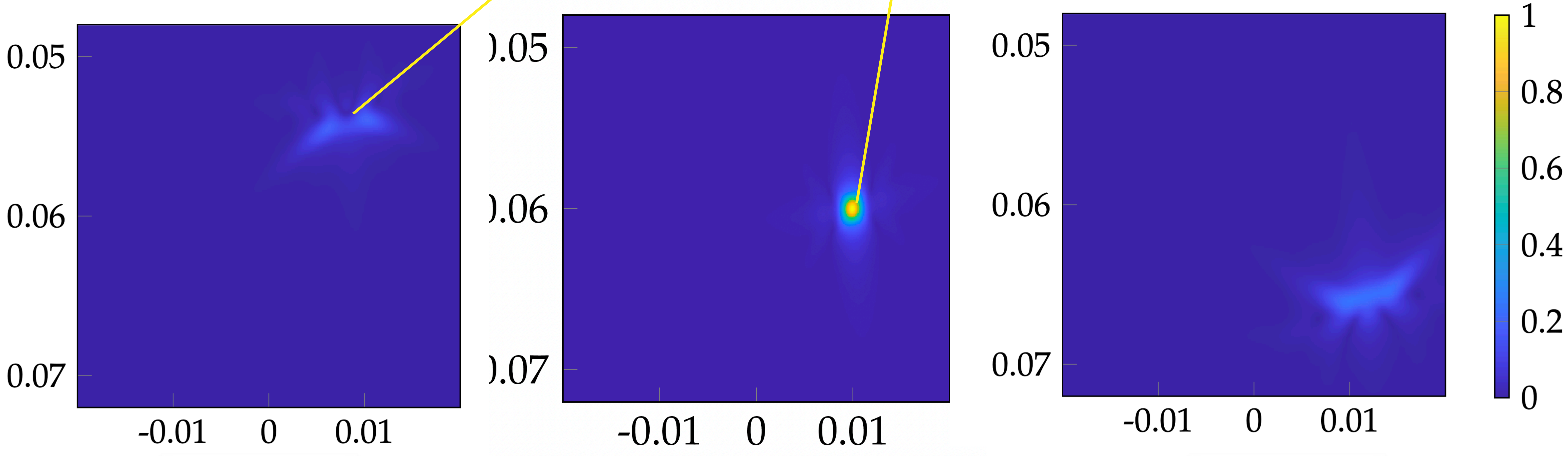
with $y_0 \in D$



Proposition : Amplitude Estimator

$$c^\star = \operatorname{argmax}_c |\mathcal{F}(c)| \,.$$

$\mathcal{F}(c_1)$ $\mathcal{F}(c_2)$



$c_1 = 0.9\,c^\star$ $c_2 = c^\star$ $c_3 = 1.1c^\star$

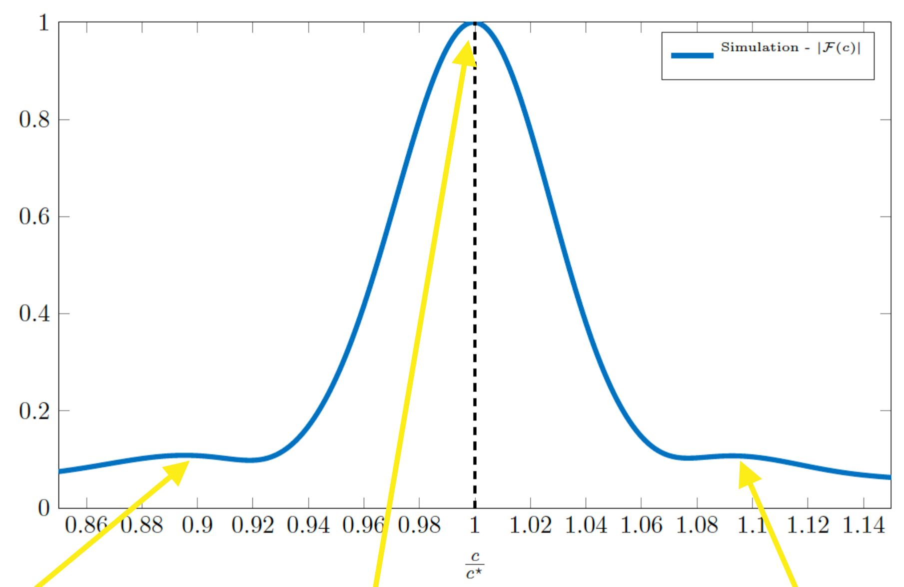
PSF amplitude estimator

Definition : Confocal $\mathscr{F}$ trace at y_0

$$[c_{\min}, c_{\max}] \longrightarrow \mathbb{C}$$

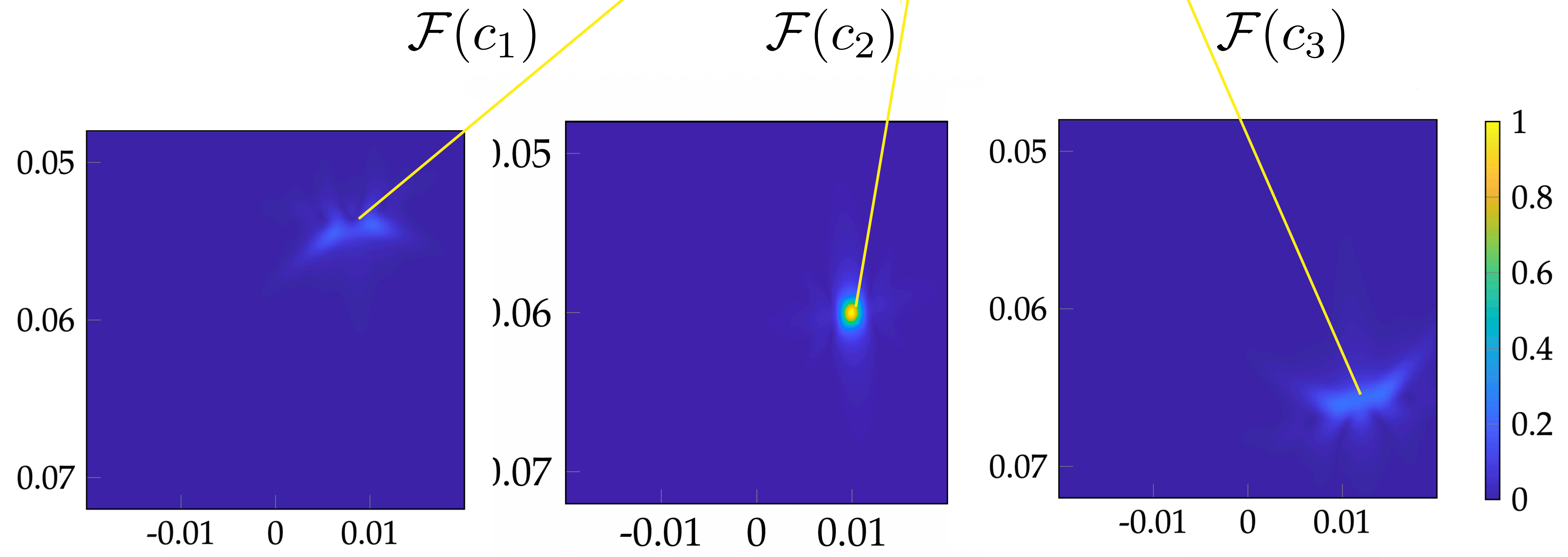
$$c \longmapsto \mathcal{F}(c) := F^{\frac{c}{c^\star}}\left(\phi_c(y_0), y_0\right)$$

with $y_0 \in D$



Proposition : Amplitude Estimator

$$c^\star = \operatorname{argmax}_c |\mathcal{F}(c)|\,.$$



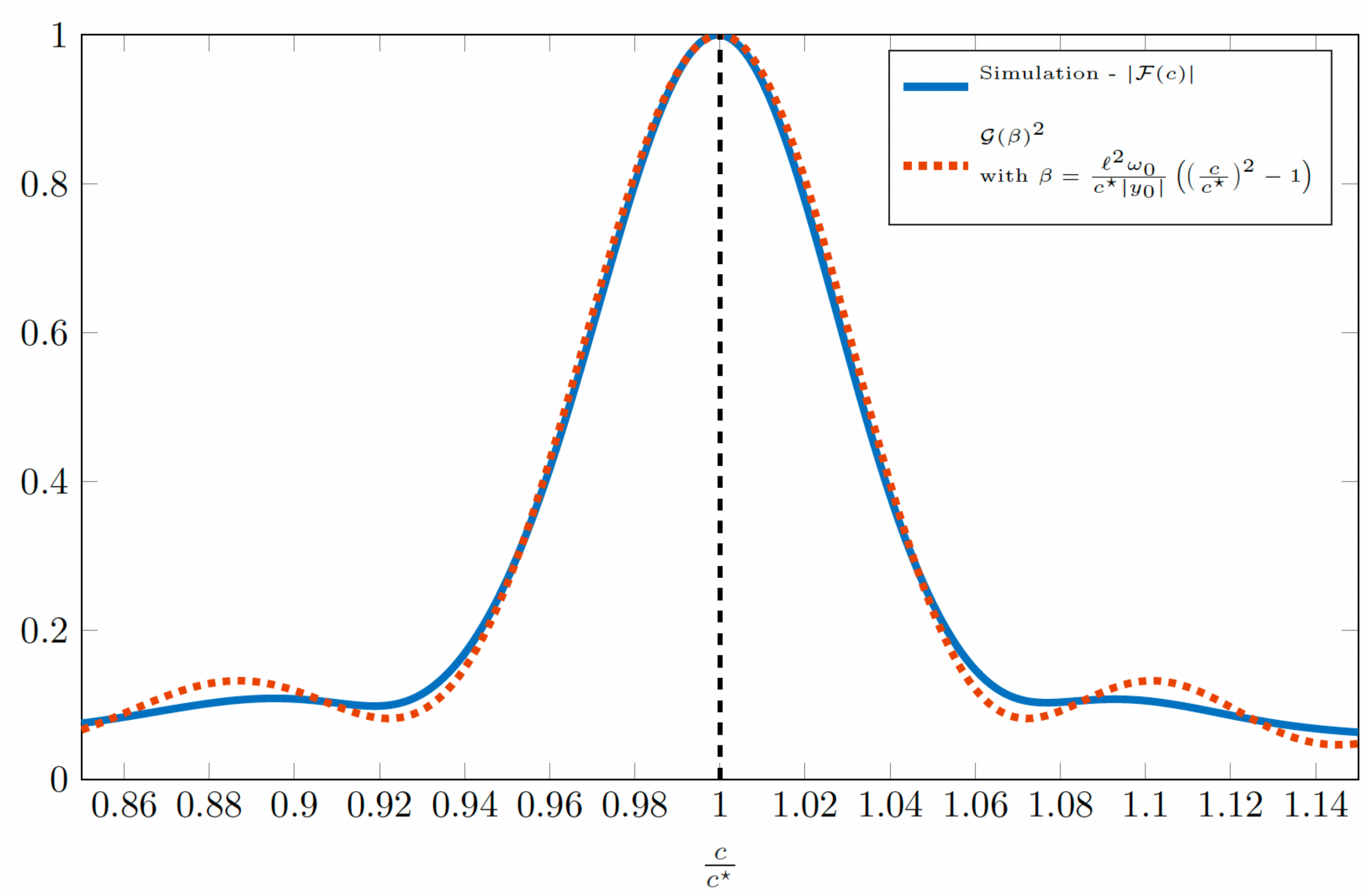
$c_1 = 0.9\,c^\star$ $c_2 = c^\star$ $c_3 = 1.1 c^\star$

Asymptotic expression

Lemma : Asymptotic development (paraxial regime)

$$\mathcal{F}(c) \sim \mathcal{G}(\beta)^2 \quad \text{with} \quad \beta = \frac{\ell^2 \omega_0}{c^\star |y_0|} \left(\left(\frac{c}{c^\star} \right)^2 - 1 \right)$$

and $$\beta \longmapsto \mathcal{G}(\beta) := \frac{1}{2\ell} \int_{\mathcal{P}} e^{i\beta \frac{x_e 2}{2}} \mathrm{d}x_e$$



Asymptotic expression

Lemma : Asymptotic development (paraxial regime)

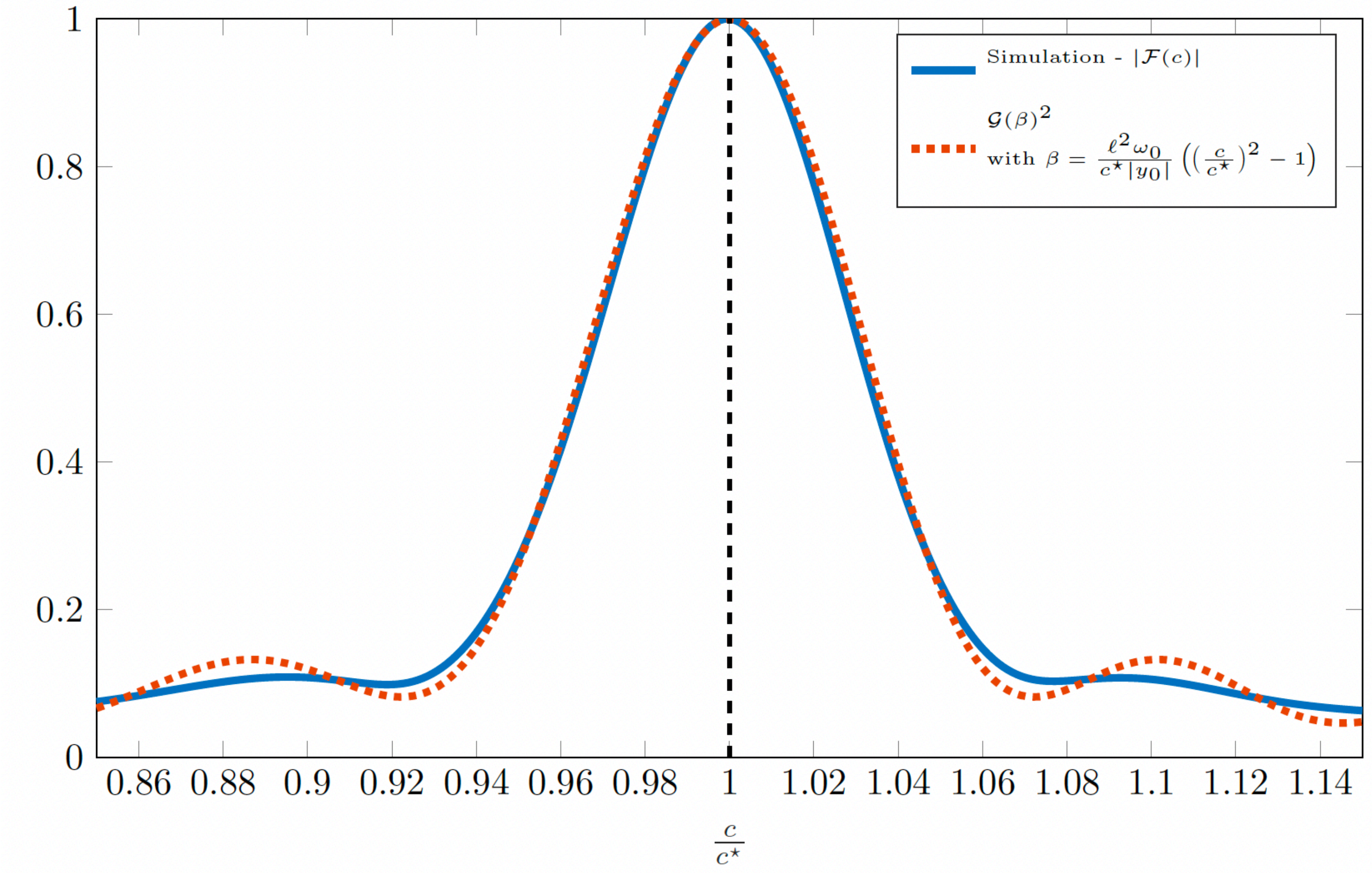
$$\mathcal{F}(c) \sim \mathcal{G}(\beta)^2 \quad \text{with} \quad \beta = \frac{\ell^2 \omega_0}{c^\star |y_0|}\left(\left(\frac{c}{c^\star}\right)^2 - 1\right)$$

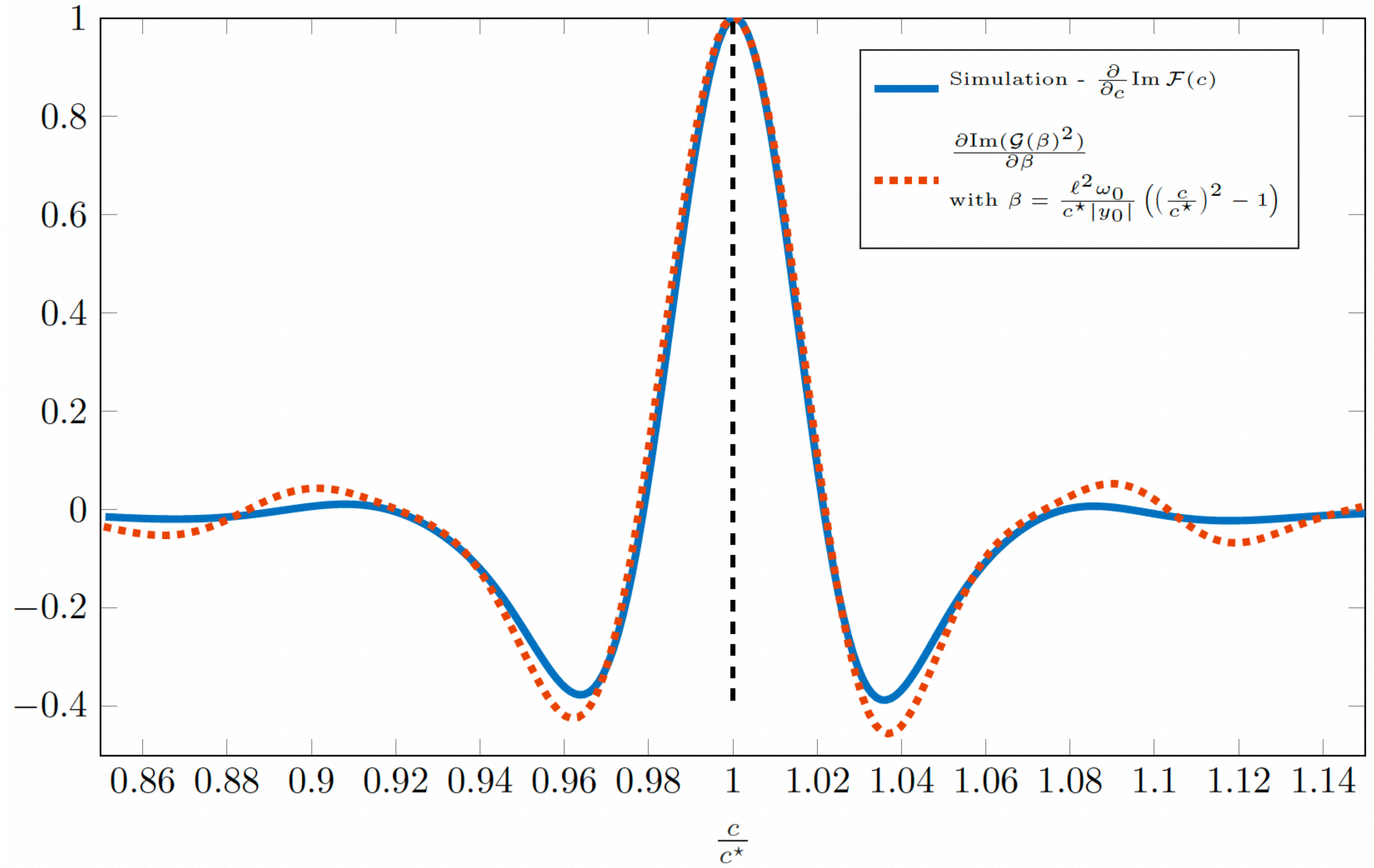
and

$$\beta \longmapsto \mathcal{G}(\beta) := \frac{1}{2\ell}\int_{\mathcal{P}} e^{i\beta \frac{x_e{}^2}{2}}\,\mathrm{d}x_e$$

Proposition : Phase jump estimator

$$c^\star = \mathrm{argmax}_c \frac{\partial}{\partial_c} \mathrm{Im}\,\mathcal{F}(c).$$





The micro-structured medium problem

- The PSF can be used to build an estimator for the speed of sound

The micro-structured medium problem

- The PSF can be used to build an estimator for the speed of sound

Maximising brightness at center of focal spot

The micro-structured medium problem

- The PSF can be used to build an estimator for the speed of sound

Maximising brightness at center of focal spot

- The PSF can be accessed on the image if there is a strong isolated small reflector in the medium (***guide star***)

The micro-structured medium problem

- The PSF can be used to build an estimator for the speed of sound

Maximising brightness at center of focal spot

- The PSF can be accessed on the image if there is a strong isolated small reflector in the medium (***guide star***)

How does one access the PSF in a microstructured medium

The dual focal spot

Image domain

$$\begin{aligned} D' &\longrightarrow D' \\ z &\longmapsto \left| F^{\frac{c}{c^\star}}(z, y_0) \right| \end{aligned}$$

Exchanging the roles of c and $c^\star$

Physical domain

$$\begin{aligned} D &\longrightarrow D \\ y &\longmapsto \left| F^{\frac{c}{c^\star}}(z_0, y) \right| \end{aligned}$$

The dual focal spot

Image domain

$$\begin{aligned} D' &\longrightarrow D' \\ z &\longmapsto \left| F^{\frac{c}{c^\star}}(z, y_0) \right| \end{aligned}$$

Exchanging the roles of c and $c^\star$

Physical domain

$$\begin{aligned} D &\longrightarrow D \\ y &\longmapsto \left| F^{\frac{c}{c^\star}}(z_0, y) \right| \end{aligned}$$

Has a small « quasi-support »

centered around $z_0(c) = \varphi_c(y_0) \in D'$

Has a small « quasi-support »

centered around $y_0(c) = \varphi_c^{-1}(z_0) \in D$

The dual focal spot

Image domain

$$\begin{aligned} D' &\longrightarrow D' \\ z &\longmapsto \left|F^{\frac{c}{c^\star}}(z, y_0)\right| \end{aligned}$$

Exchanging the roles of c and $c^\star$

Physical domain

$$\begin{aligned} D &\longrightarrow D \\ y &\longmapsto \left|F^{\frac{c}{c^\star}}(z_0, y)\right| \end{aligned}$$

Has a small « quasi-support »

centered around $z_0(c) = \varphi_c(y_0) \in D'$

The focal spot

The dual focal spot

Has a small « quasi-support »

centered around $y_0(c) = \varphi_c^{-1}(z_0) \in D$

The dual focal spot

Image domain

$$\begin{aligned} D' &\longrightarrow D' \\ z &\longmapsto \left|F^{\frac{c}{c^\star}}(z, y_0)\right| \end{aligned}$$

Exchanging the roles of c and $c^\star$

Physical domain

$$\begin{aligned} D &\longrightarrow D \\ y &\longmapsto \left|F^{\frac{c}{c^\star}}(z_0, y)\right| \end{aligned}$$

Has a small « quasi-support »

centered around $z_0(c) = \varphi_c(y_0) \in D'$

The focal spot

The dual focal spot

Has a small « quasi-support »

centered around $y_0(c) = \varphi_c^{-1}(z_0) \in D$

$D'_\delta(c, y_0)$

$D_\delta(c, z_0)$

The part of the image affected by the point $y_0 \in D$

The part of the domain probed by the pixel $z_0 \in D'$

Virtual random reflector

$$I^c(z) = \int_D (n(y) - n^\star)\, F^{\frac{c}{c^\star}}(z,y)\mathrm{d}y, \qquad z \in D' \quad \longrightarrow \quad I^c(z) \approx \int_{D(c,z)} (n(y) - n^\star)\, F^{\frac{c}{c^\star}}(z,y)\mathrm{d}y, \qquad z \in D'$$

Virtual random reflector

$$I^c(z) = \int_D (n(y) - n^\star)\, F^{\frac{c}{c^\star}}(z,y)\mathrm{d}y, \qquad z \in D' \quad \longrightarrow \quad I^c(z) \approx \int_{D(c,z)} (n(y) - n^\star)\, F^{\frac{c}{c^\star}}(z,y)\mathrm{d}y, \qquad z \in D'$$

Whole physical domain

Virtual random reflector

$$I^c(z) = \int_D (n(y) - n^\star)\, F^{\frac{c}{c^\star}}(z,y)\mathrm{d}y, \qquad z \in D' \longrightarrow I^c(z) \approx \int_{\color{red}{D(c,z)}} (n(y) - n^\star)\, F^{\frac{c}{c^\star}}(z,y)\mathrm{d}y, \qquad z \in D'$$

Whole physical domain

Small domain around $\varphi_c^{-1}(z)$

Virtual random reflector

$$I^c(z) = \int_D (n(y) - n^\star)\, F^{\frac{c}{c^\star}}(z,y)\mathrm{d}y, \qquad z \in D' \longrightarrow I^c(z) \approx \int_{D(c,z)} (n(y) - n^\star)\, F^{\frac{c}{c^\star}}(z,y)\mathrm{d}y, \qquad z \in D'$$

Whole physical domain

Small domain around $\varphi_c^{-1}(z)$

$I^c(z)$ depends on the realization of $n(y) - n^\star$ in the small subdomain $D(c,z) \subset D$

Virtual random reflector

$$I^c(z) = \int_D (n(y) - n^\star)\, F^{\frac{c}{c^\star}}(z,y)\mathrm{d}y, \qquad z \in D' \longrightarrow I^c(z) \approx \int_{D(c,z)} (n(y) - n^\star)\, F^{\frac{c}{c^\star}}(z,y)\mathrm{d}y, \qquad z \in D'$$

Whole physical domain

Small domain around $\varphi_c^{-1}(z)$

$I^c(z)$ depends on the realization of $n(y) - n^\star$ in the small subdomain $D(c,z) \subset D$

Proposition : Variance of I^c

$$\mathbb{E}\left[\left|I^c(z)\right|^2\right] = \varepsilon^d C_n \int_{D(c,z)} \left|F^c(z,y)\right|^2 \mathrm{d}y + o\left(\varepsilon^d\right)$$

C_n the integral of the covariance of n.

Virtual random reflector

$$I^c(z) = \int_D (n(y) - n^\star)\, F^{\frac{c}{c^\star}}(z,y)\mathrm{d}y, \qquad z \in D' \longrightarrow I^c(z) \approx \int_{D(c,z)} (n(y) - n^\star)\, F^{\frac{c}{c^\star}}(z,y)\mathrm{d}y, \qquad z \in D'$$

Whole physical domain

Small domain around $\varphi_c^{-1}(z)$

$I^c(z)$ depends on the realization of $n(y) - n^\star$ in the small subdomain $D(c,z) \subset D$

Proposition : Variance of I^c

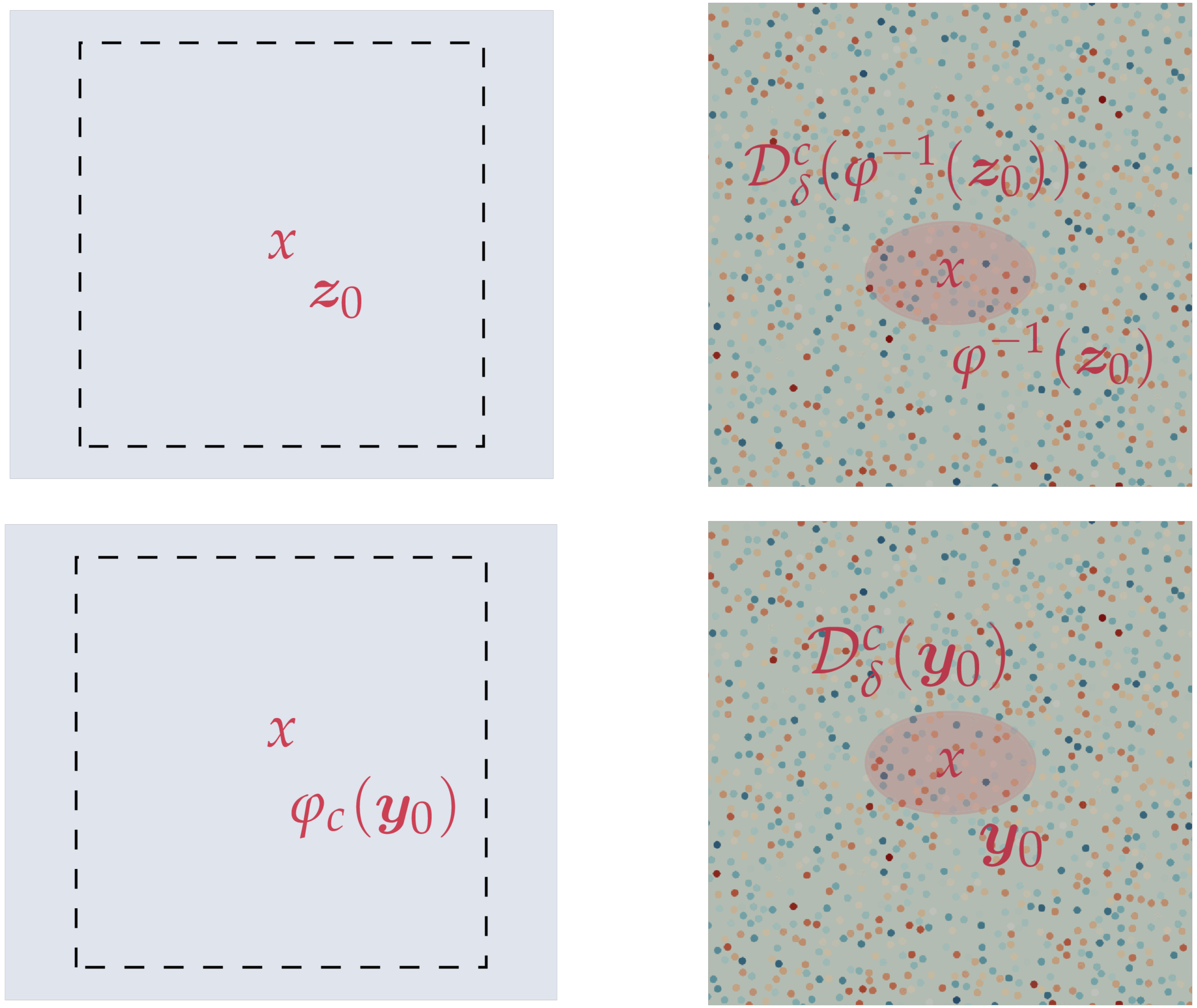
$$\mathbb{E}\left[\left| I^c(z) \right|^2 \right] = \varepsilon^d C_n \int_{D(c,z)} \left| F^c(z,y) \right|^2 \mathrm{d}y + o\left(\varepsilon^d\right)$$

C_n the integral of the covariance of n.

D(c,z) centered around $\varphi_c^{-1}(z)$ - moves with c

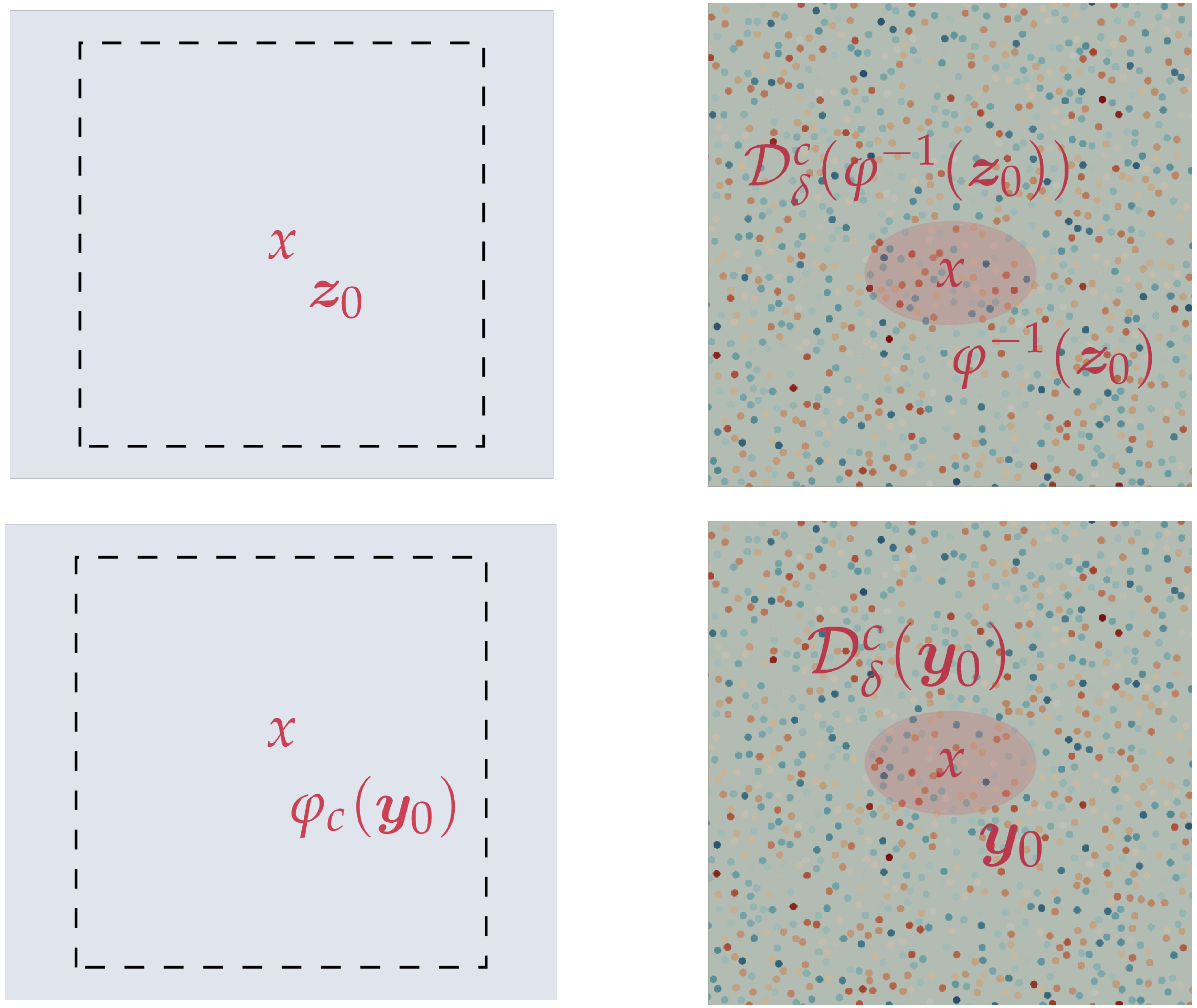
Moving focal spots

D' (Image) $c = 1.2 * c^\star$ D (Physical domain)



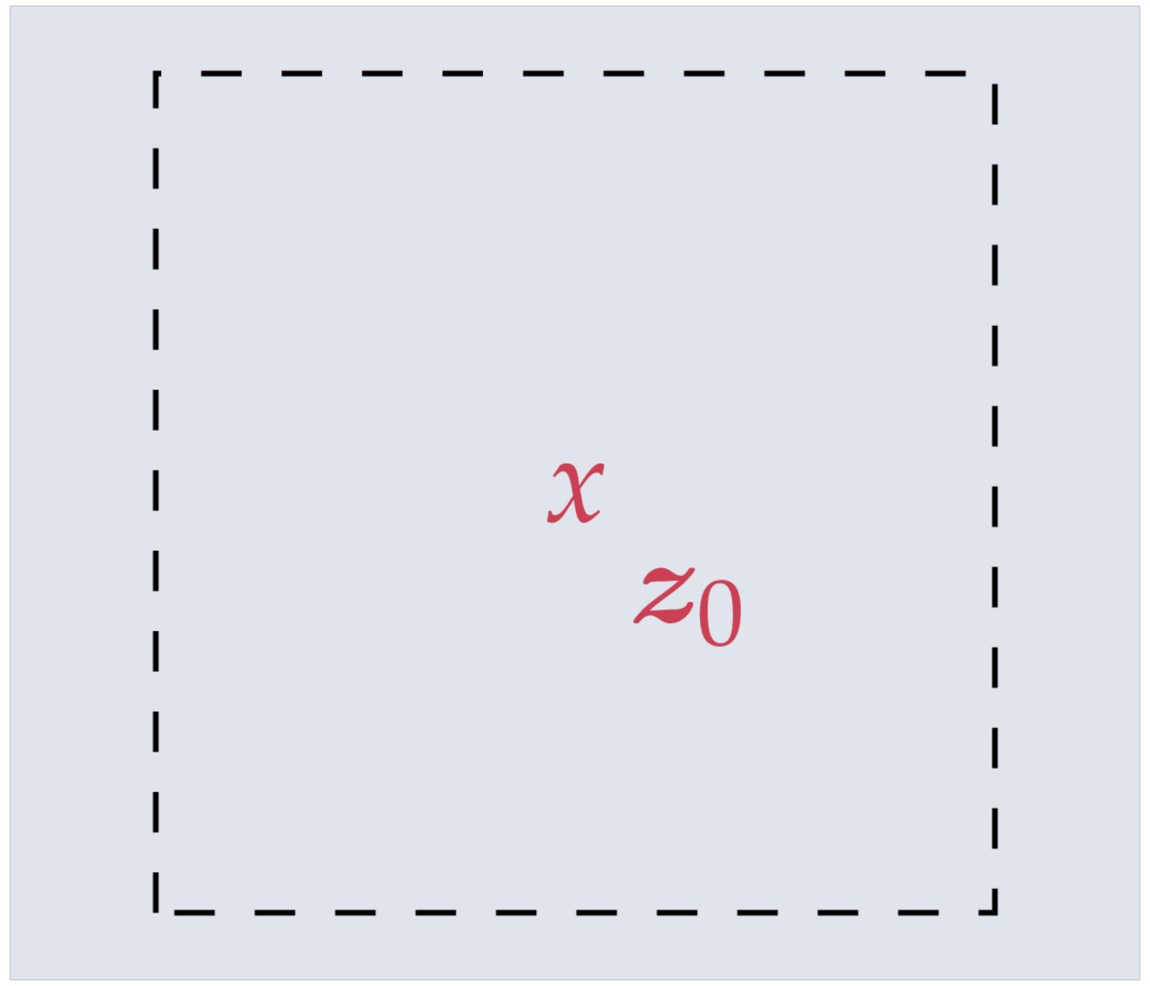
Moving focal spots

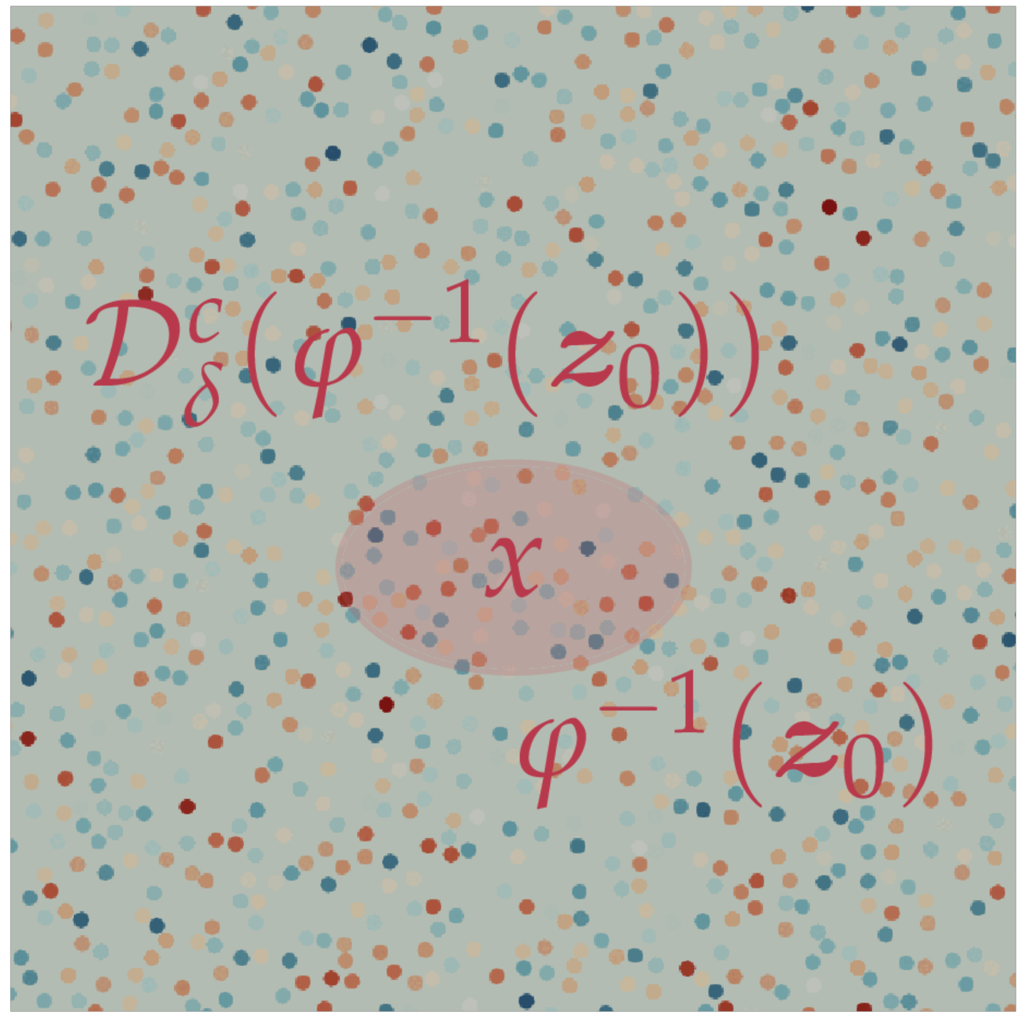
D' (Image) $\quad c = 1.2 * c^{\star} \quad D$ (Physical domain)

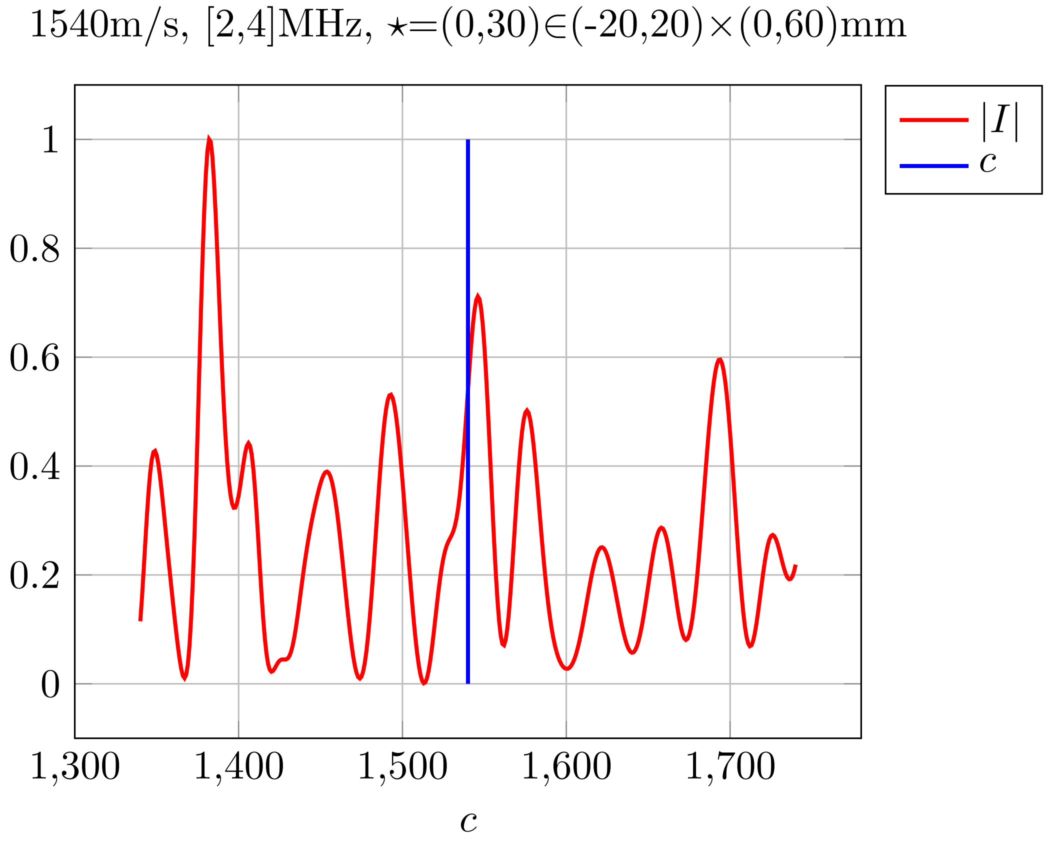


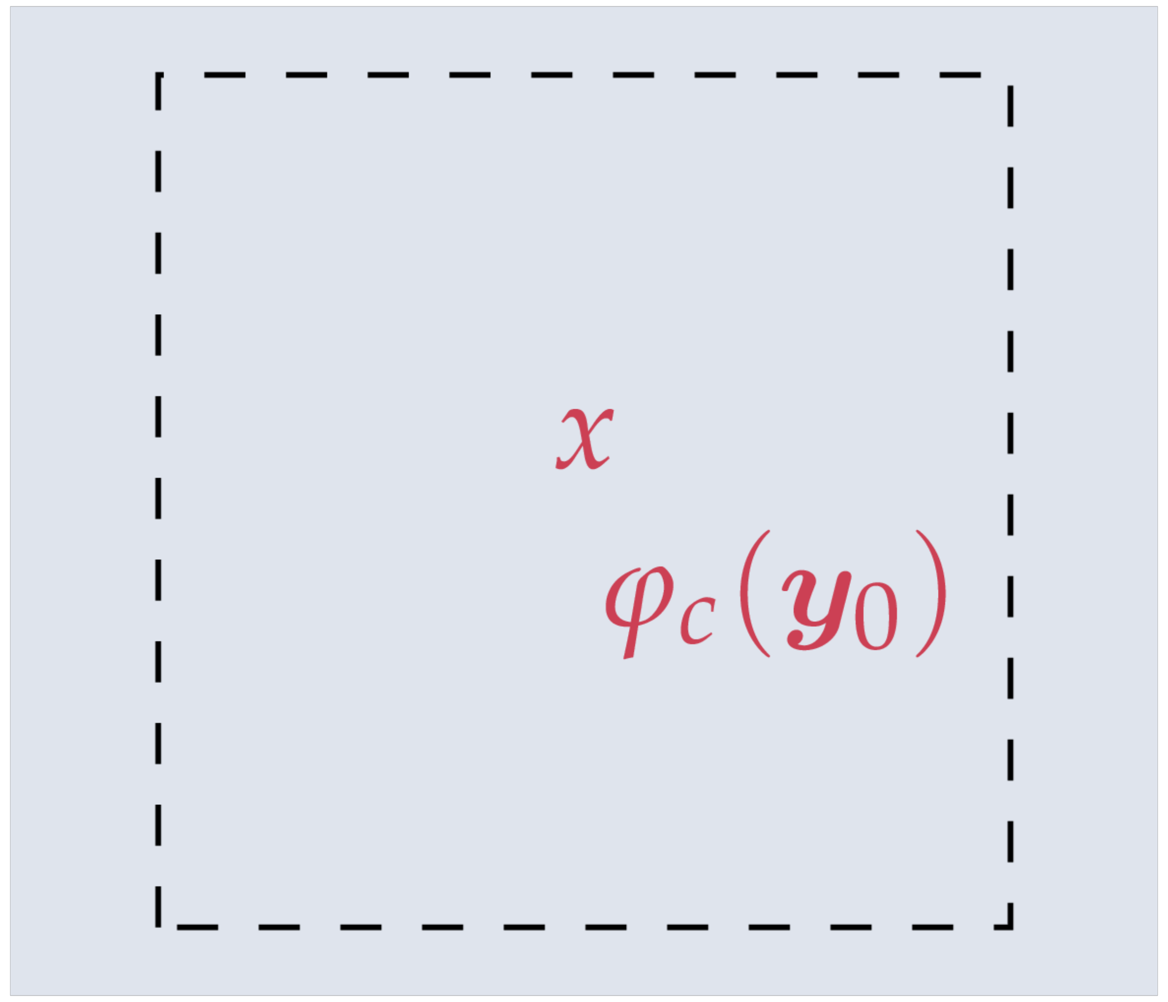
Moving focal spots

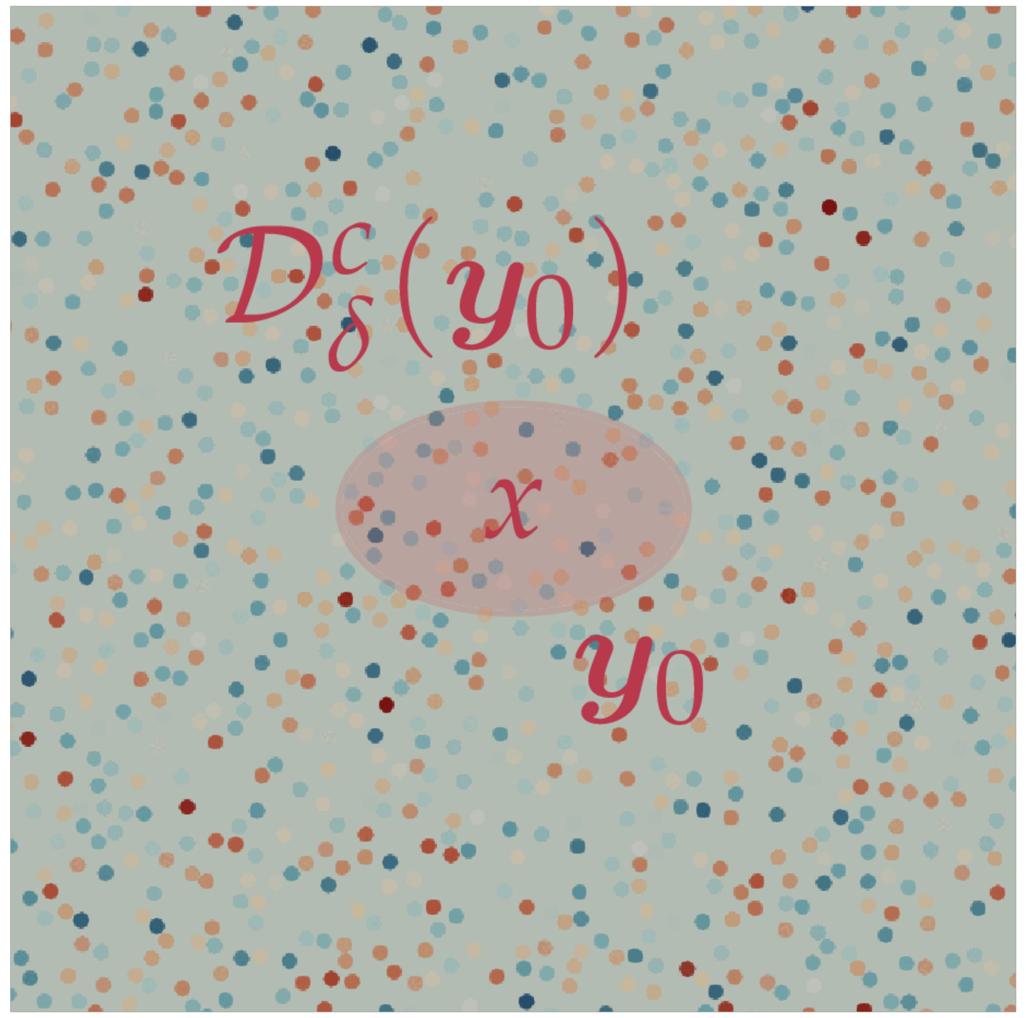
D' (Image) $c = 1.2 * c^\star$ D (Physical domain) $c \mapsto |I^c(z)|^2$



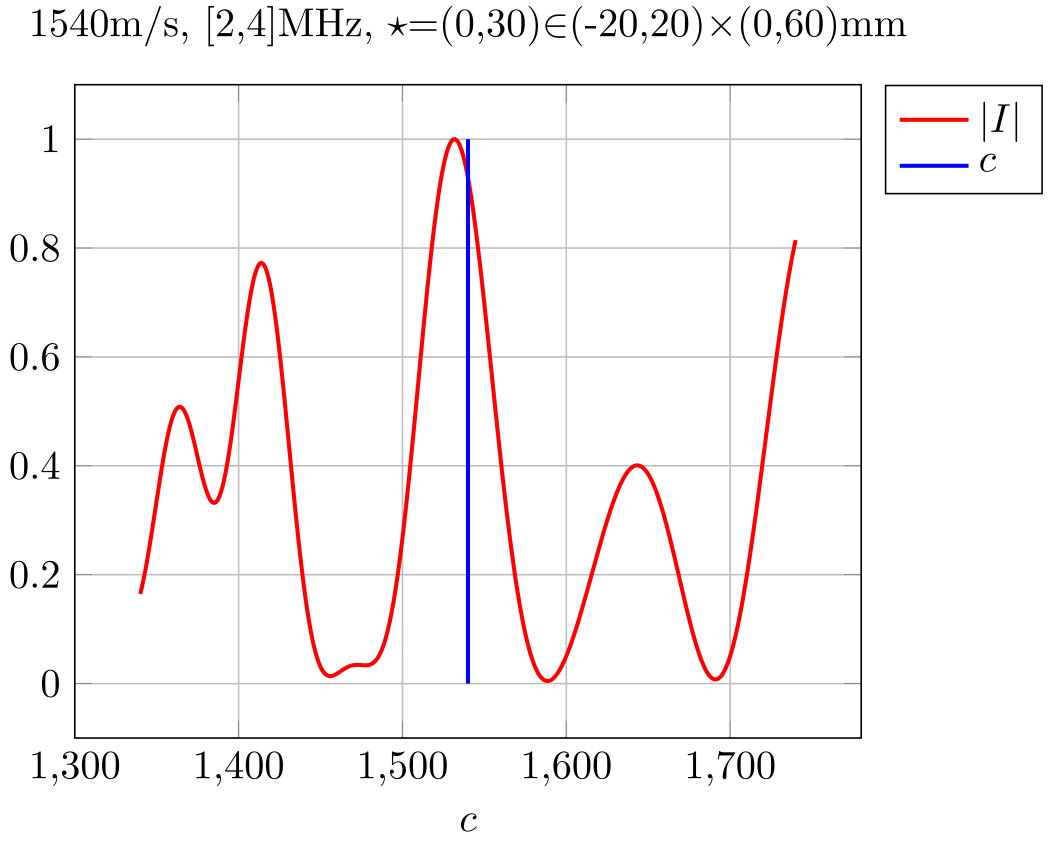








$c \mapsto |I^c(\varphi_c(y_0))|^2$

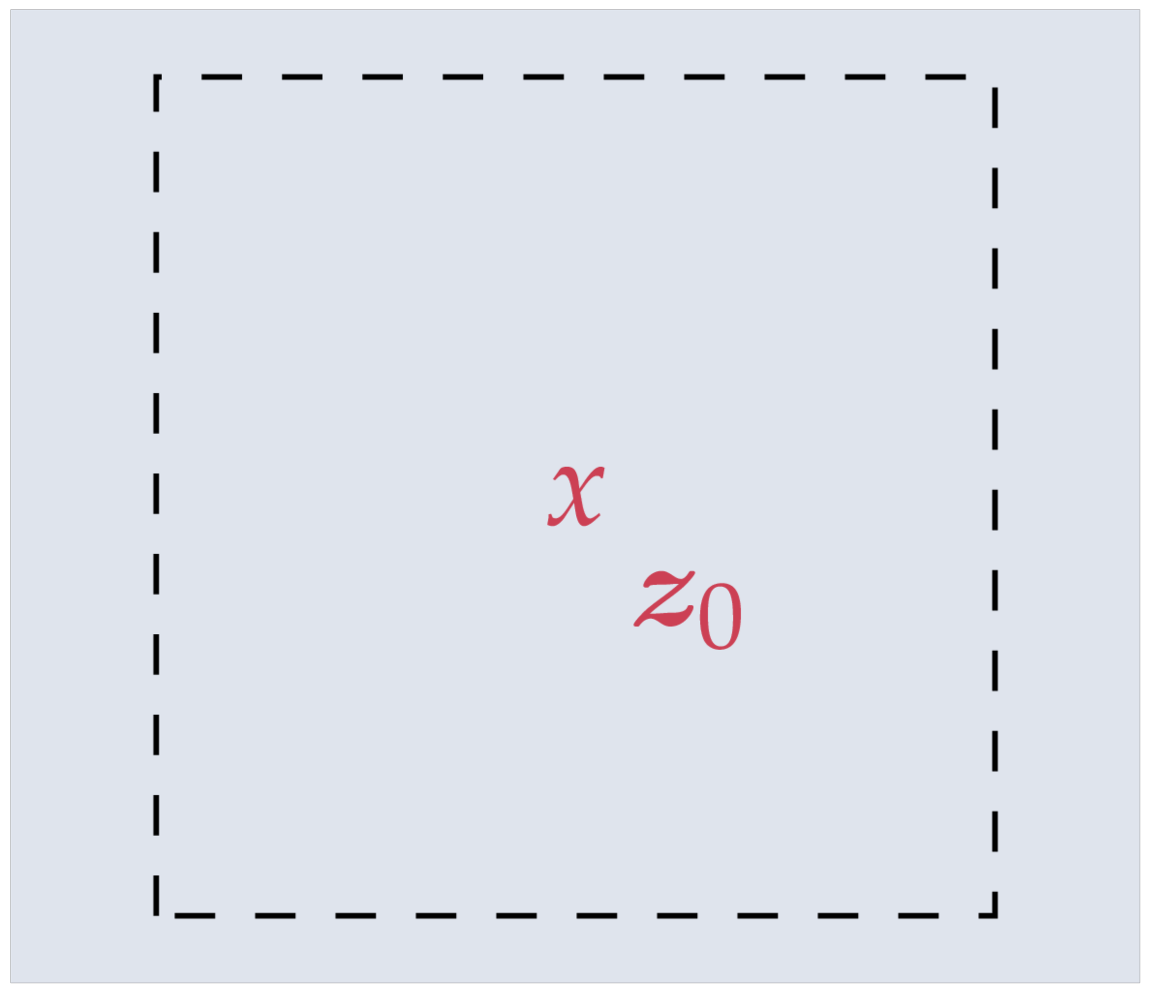


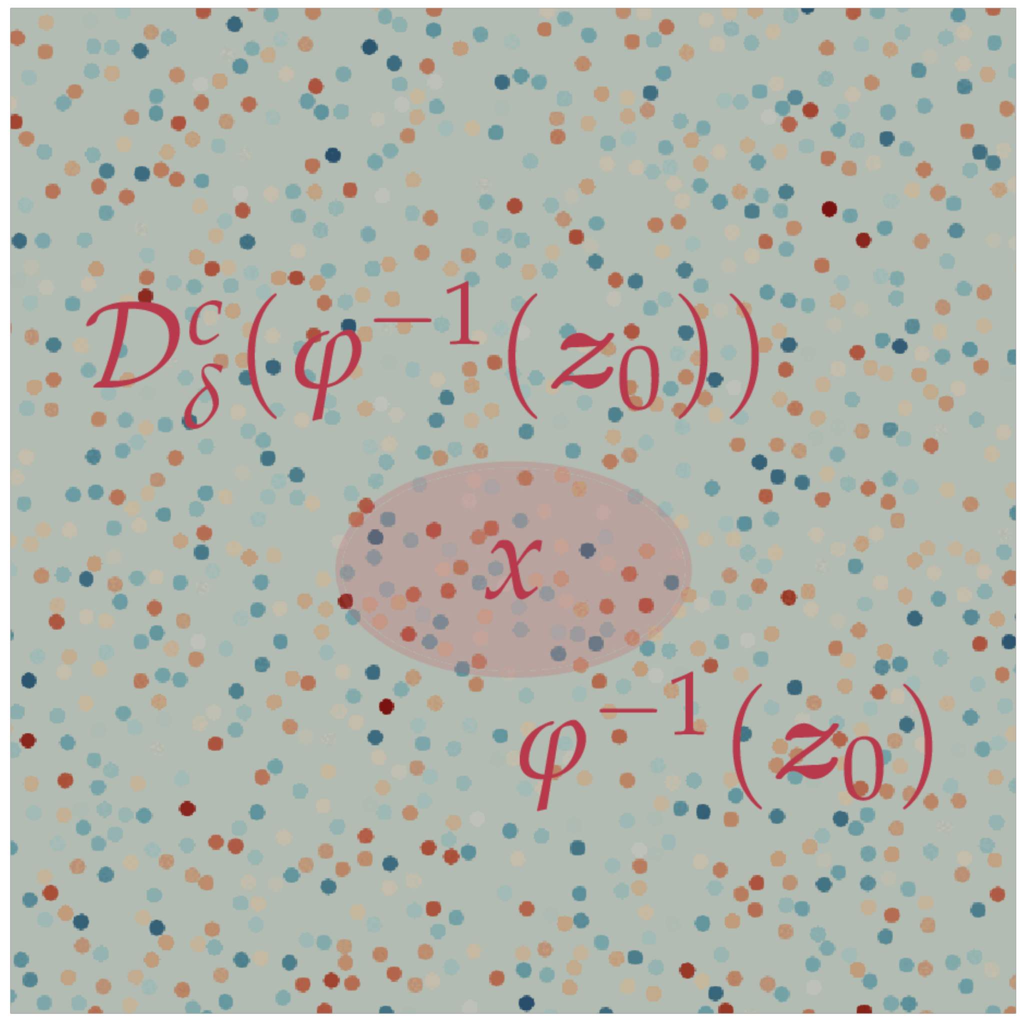
Moving focal spots

1 realization

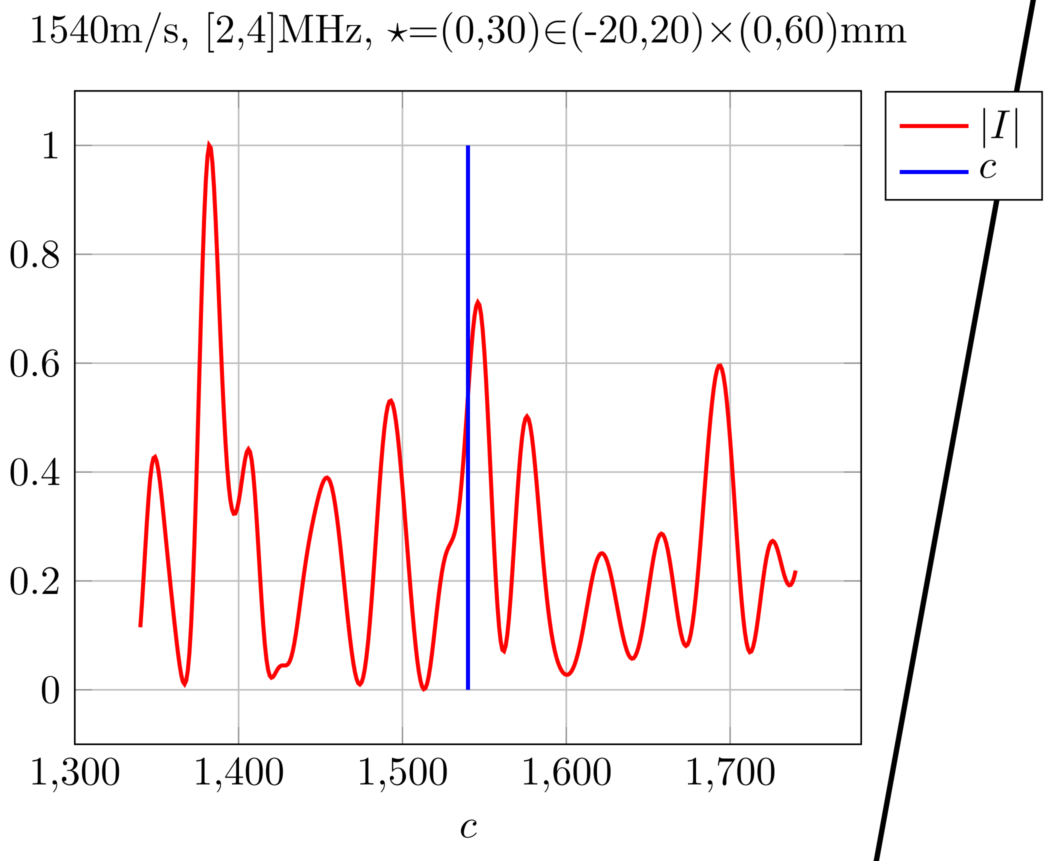
20 realizations

D' (Image) $\quad c = 1.2 * c^\star \quad D$ (Physical domain)

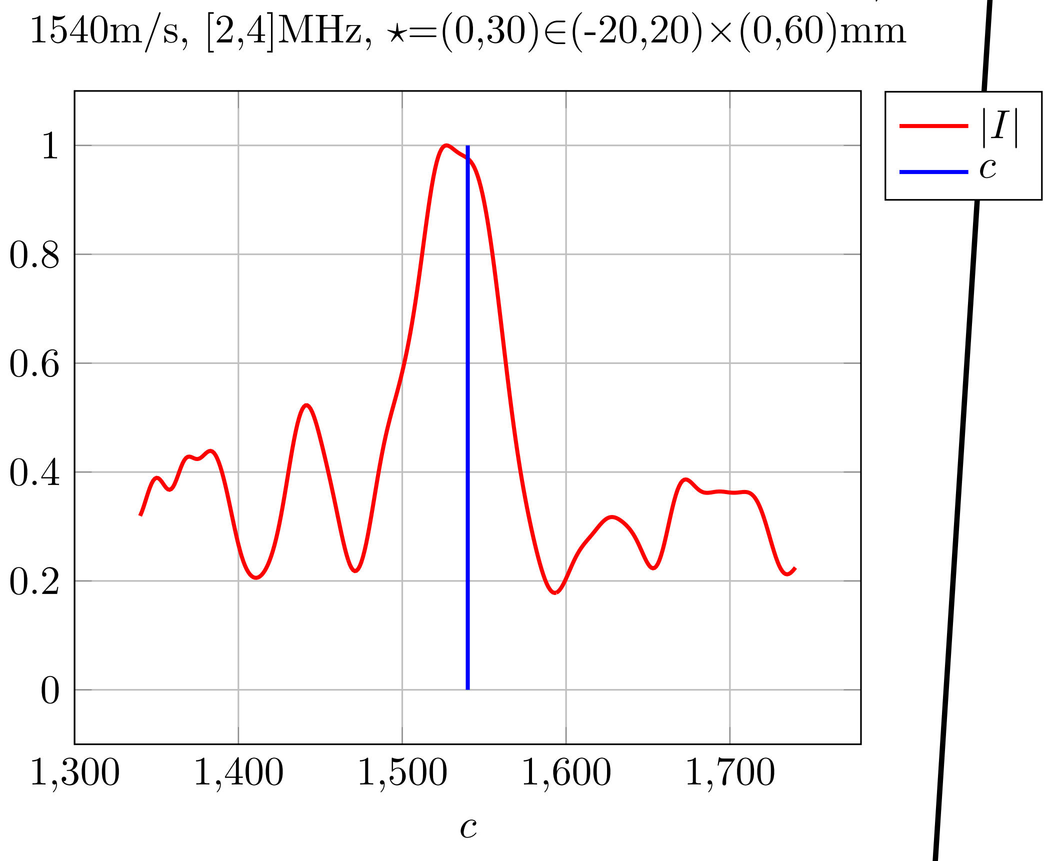


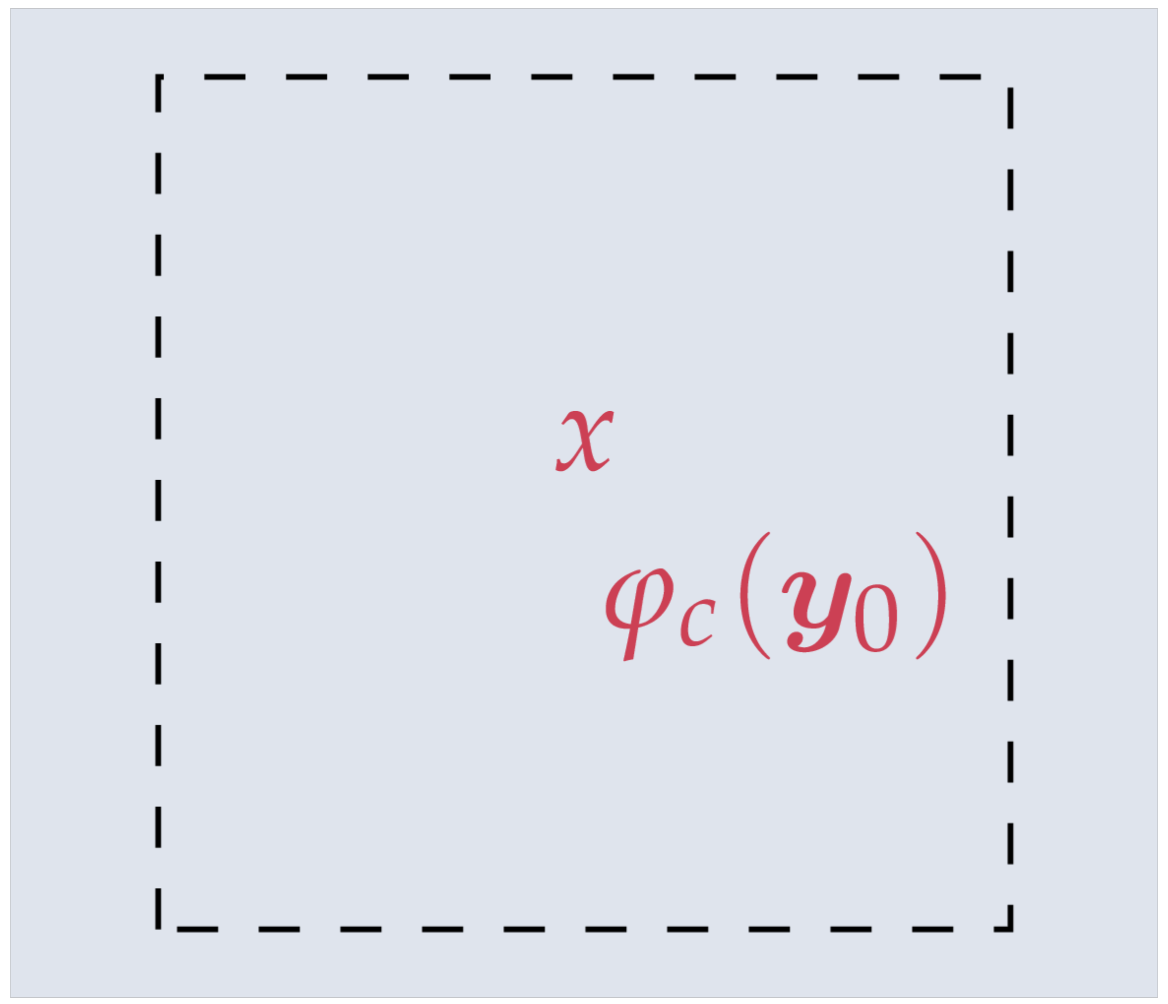


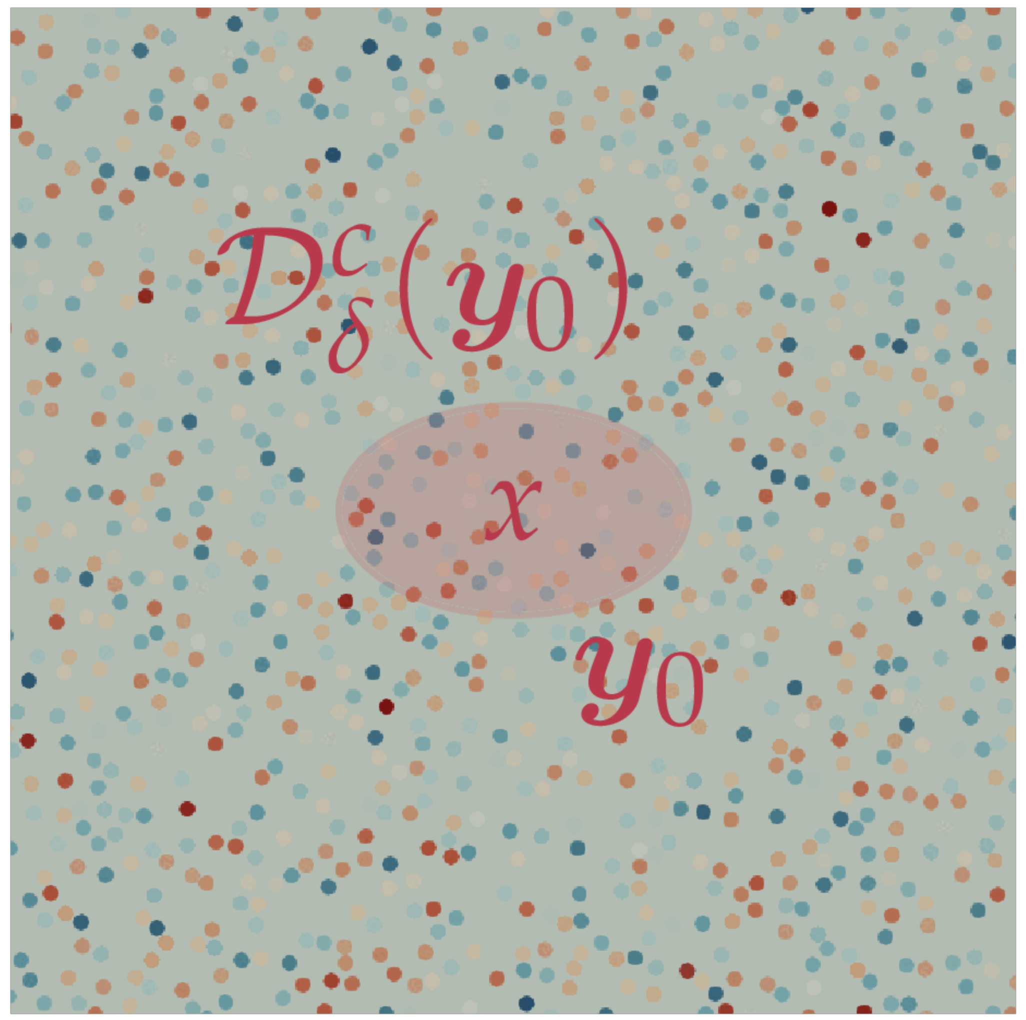
$c \mapsto |I^c(z)|^2$



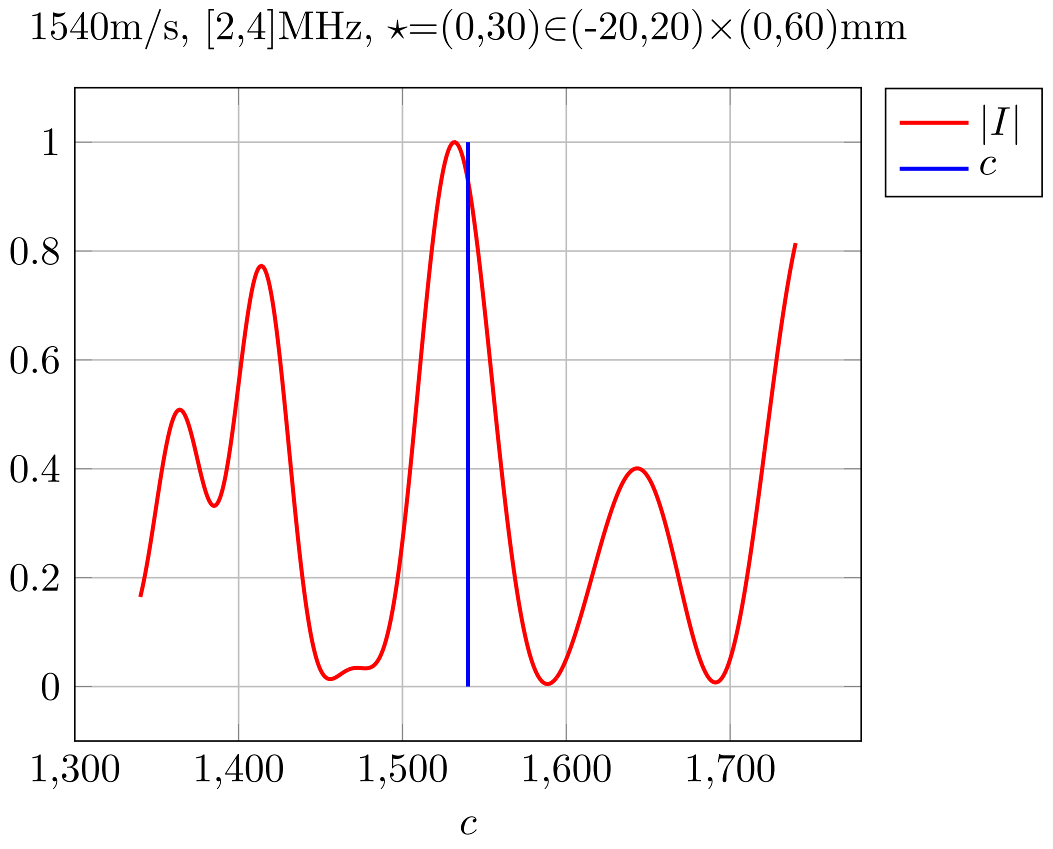
$c \mapsto \mathbb{E}\left[|I^c(z)|^2\right]$



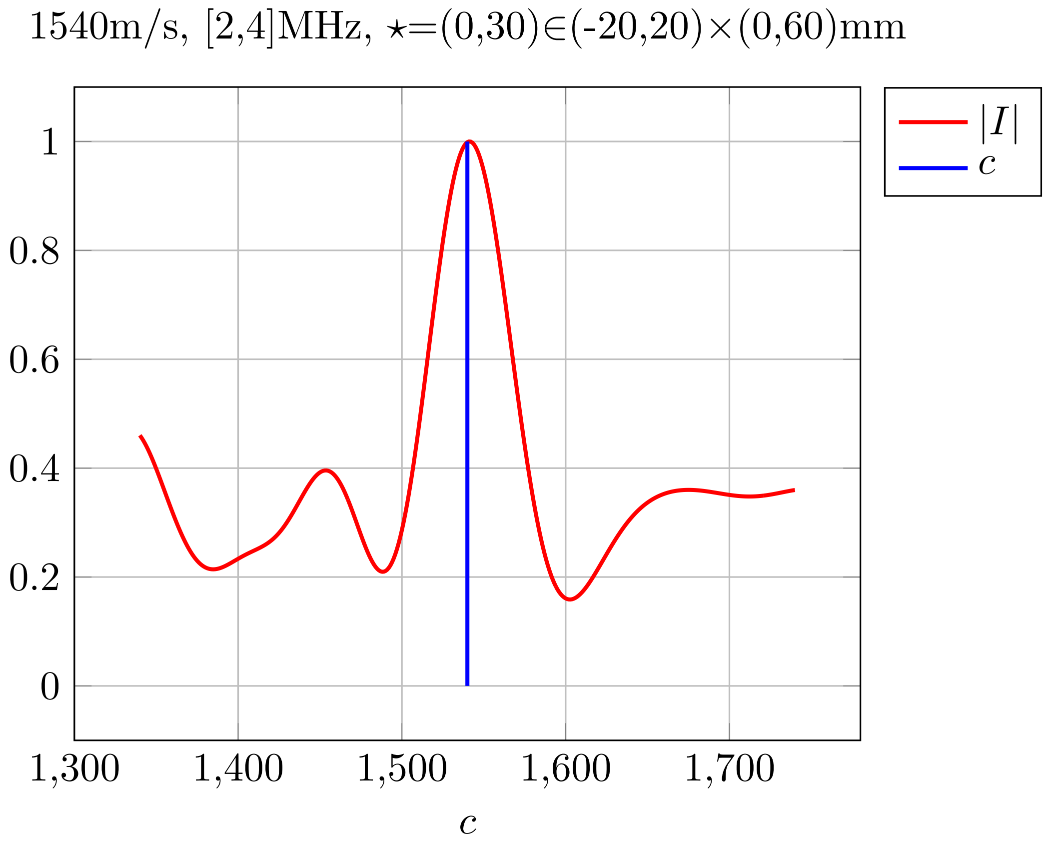




$c \mapsto |I^c(\varphi_c(y_0))|^2$



$c \mapsto \mathbb{E}\left[|I^c(\varphi_c(y_0))|^2\right]$



Fixing the focal spot

Pick $t_0 \gg \omega_0^{-1}$ and consider $y_{t_0} := (0, c^\star t_0)$

UNKNOWN point in the domain, only t_0 known

$$\varphi_c(y_{t_0}) = (0, \frac{c}{c^\star} y_{t_0}^{\parallel}) = (0, ct_0)$$

Only depends on t_0 and c, KNOWN

Fixing the focal spot

Pick $t_0 \gg \omega_0^{-1}$ and consider $y_{t_0} := (0, c^\star t_0)$

UNKNOWN point in the domain, only t_0 known

$$\varphi_c(y_{t_0}) = (0, \frac{c}{c^\star} y^{\parallel}_{t_0}) = (0, ct_0)$$

Only depends on t_0 and c, KNOWN

Theorem: Incoherent estimator

$$c^\star = \mathrm{argmax}_c \, \mathbb{E}\left[I^c(z(c(t_0)))^2\right] \quad \text{with} \quad z(c) = (0, ct_0)$$

Fixing the focal spot

Pick $t_0 \gg \omega_0^{-1}$ and consider $y_{t_0} := (0, c^\star t_0)$ UNKNOWN point in the domain, only t_0 known

$$\varphi_c(y_{t_0}) = (0, \frac{c}{c^\star} y_{t_0}^{\shortparallel}) = (0, ct_0)$$ Only depends on t_0 and c, KNOWN

Theorem: Incoherent estimator

$$c^\star = \operatorname{argmax}_c \mathbb{E}\left[I^c(z(c(t_0)))^2\right] \quad \text{with} \quad z(c) = (0, ct_0)$$

$$\mathbb{E}\left[|I^c(z(t_0))|^2\right] = \varepsilon^d C_n \int_{D^c(y_0)} |F^c(z,y)|^2 \,\mathrm{d}y + o\left(\varepsilon^d\right)$$

Dual focal spot centered on y_{t_0}

Point spread function

Fixing the focal spot

Pick $t_0 \gg \omega_0^{-1}$ and consider $y_{t_0} := (0, c^\star t_0)$ UNKNOWN point in the domain, only t_0 known

$$\varphi_c(y_{t_0}) = (0, \frac{c}{c^\star} y_{t_0}^{\shortparallel}) = (0, ct_0)$$

Only depends on t_0 and c, KNOWN

Theorem: Incoherent estimator

$c^\star = \text{argmax}_c \, \mathbb{E}\left[I^c(z(c(t_0)))^2\right]$ with $z(c) = (0, ct_0)$

Only one realization of the medium

$$\mathbb{E}\left[\left|I^c(z(t_0))\right|^2\right] = \varepsilon^d C_n \int_{D^c(y_0)} \left|F^c(z,y)\right|^2 \mathrm{d}y + o\left(\varepsilon^d\right)$$

Dual focal spot centered on y_{t_0}

Point spread function

Ensemble averaging via ergodicity

Denote by $n(\cdot,\gamma), \gamma \in \Omega$ the dependency on the randomness of the medium. Consider $\varepsilon \ll |\Delta z| \ll \dfrac{c}{\omega_0} \sim \dfrac{c^\star}{\omega_0}$.

Ensemble averaging via ergodicity

Denote by $n(\cdot,\gamma), \gamma \in \Omega$ the dependency on the randomness of the medium. Consider $\varepsilon \ll |\Delta z| \ll \frac{c}{\omega_0} \sim \frac{c^\star}{\omega_0}$.

Lemma: Stationarity of the medium

$$n(\cdot,\gamma)\big|_{D_\eta(c,z)} = n(\cdot,\tau_{\Delta z}\gamma)\big|_{D_\eta(c,z+\Delta z)}$$

Ensemble averaging via ergodicity

Denote by $n(\cdot,\gamma), \gamma \in \Omega$ the dependency on the randomness of the medium. Consider $\varepsilon \ll |\Delta z| \ll \dfrac{c}{\omega_0} \sim \dfrac{c^\star}{\omega_0}$.

Lemma: Stationarity of the medium

$$n(\cdot,\gamma)\big|_{D_\eta(c,z)} = n(\cdot,\tau_{\Delta z}\gamma)\big|_{D_\eta(c,z+\Delta z)}$$

Slightly shifted dual focal spot of z

Dual focal spot of z

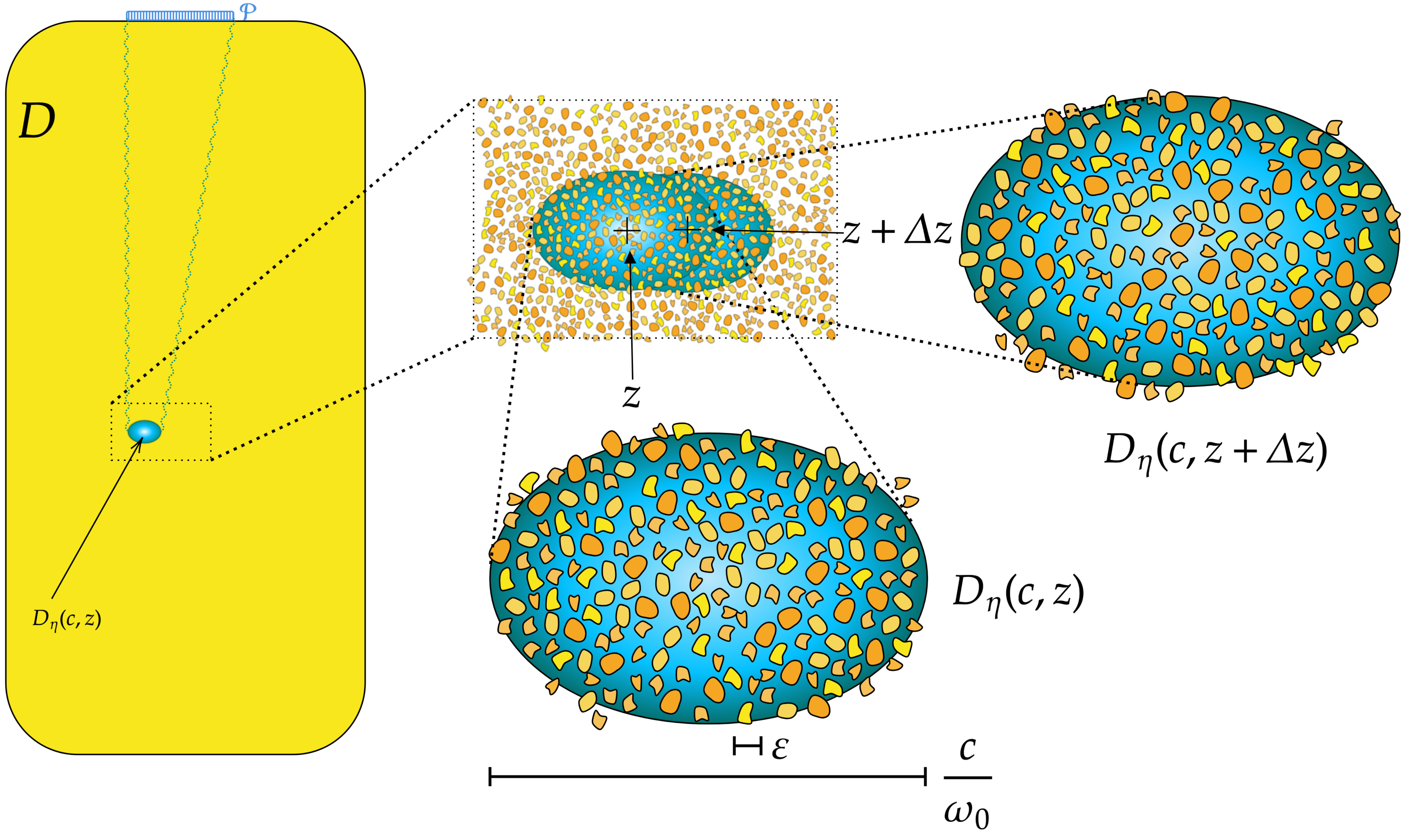
Group action defining the ergodicity of the process

Ensemble averaging via ergodicity

Denote by $n(\cdot,\gamma), \gamma \in \Omega$ the dependency on the randomness of the medium. Consider $\varepsilon \ll |\Delta z| \ll \dfrac{c}{\omega_0} \sim \dfrac{c^\star}{\omega_0}$.

Lemma: Stationarity of the medium

$$n(\cdot,\gamma)\big|_{D_\eta(c,z)} = n(\cdot,\tau_{\Delta z}\gamma)\big|_{D_\eta(c,z+\Delta z)}$$

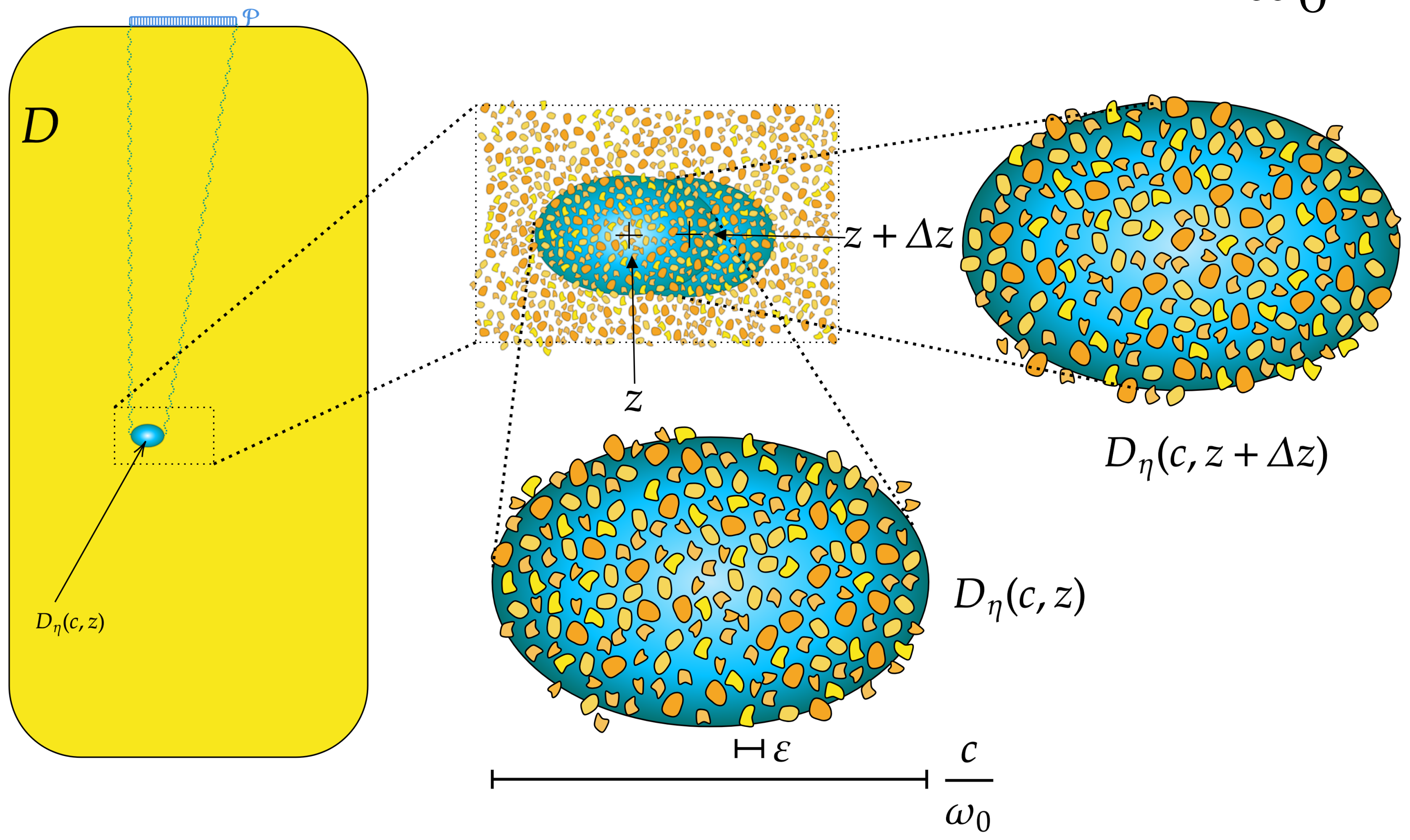


Ensemble averaging via ergodicity

Denote by $n(\cdot,\gamma), \gamma \in \Omega$ the dependency on the randomness of the medium. Consider $\varepsilon \ll |\Delta z| \ll \dfrac{c}{\omega_0} \sim \dfrac{c^\star}{\omega_0}$.

Lemma: Stationarity of the medium

$$n(\cdot,\gamma)\big|_{D_\eta(c,z)} = n(\cdot,\tau_{\Delta z}\gamma)\big|_{D_\eta(c,z+\Delta z)}$$



Proposition : Ergodicity of I^c

For $\Delta z \in \mathbb{R}^d$ such that $\varepsilon \ll |\Delta z| \ll \dfrac{c^\star}{\omega_0}$, there exists $\gamma' \in \Omega$ such that :

$$I^c(z+\Delta z,\gamma) = I^c(z,\gamma') + o\left(\frac{c^\star}{\omega_0}\right)$$

Ensemble averaging via ergodicity

Denote by $n(\cdot,\gamma), \gamma \in \Omega$ the dependency on the randomness of the medium. Consider $\varepsilon \ll |\Delta z| \ll \frac{c}{\omega_0} \sim \frac{c^\star}{\omega_0}$.

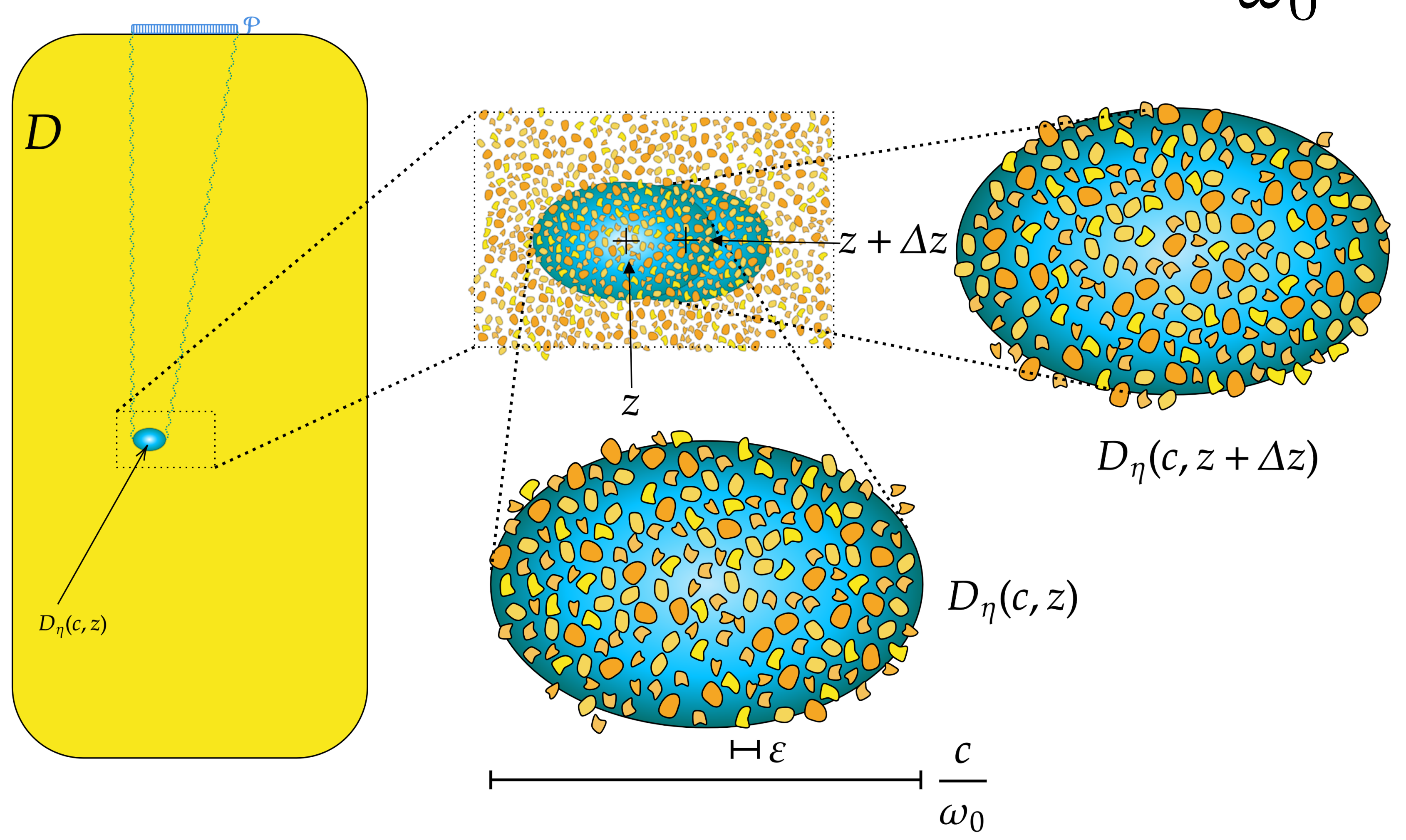
Lemma: Stationarity of the medium

$$n(\cdot,\gamma)\big|_{D_\eta(c,z)} = n(\cdot,\tau_{\Delta z}\gamma)\big|_{D_\eta(c,z+\Delta z)}$$

Proposition : Ergodicity of I^c

For $\Delta z \in \mathbb{R}^d$ such that $\varepsilon \ll |\Delta z| \ll \frac{c^\star}{\omega_0}$, there exists $\gamma' \in \Omega$ such that :

$$I^c(z+\Delta z,\gamma) = I^c(z,\gamma') + o\left(\frac{c^\star}{\omega_0}\right)$$



Possible **in practice** to access statistics of I^c w/ **one realization**

Building the estimator in practice

Pick a travel time $t_0 \gg \omega_0^{-1}$.

Building the estimator in practice

Pick a travel time $t_0 \gg \omega_0^{-1}$.

Define the set of possible values for c : $[c_{min}, c_{max}]$. Define the probing points $z(c) = (0, ct_0) \quad \forall c \in [c_{min}, c_{max}]$.

Building the estimator in practice

Pick a travel time $t_0 \gg \omega_0^{-1}$.

Define the set of possible values for c : $[c_{min}, c_{max}]$. Define the probing points $z(c) = (0, ct_0) \qquad \forall c \in [c_{min}, c_{max}]$.

Define a set $\mathcal{S}$ of small spatial shifts Δz satisfying $\varepsilon \ll |\Delta z| \ll \dfrac{c_{min}}{\omega_0}$.

Building the estimator in practice

Pick a travel time $t_0 \gg \omega_0^{-1}$.

Define the set of possible values for c : $[c_{min}, c_{max}]$. Define the probing points $z(c) = (0, ct_0) \qquad \forall c \in [c_{min}, c_{max}]$.

Define a set $\mathcal{S}$ of small spatial shifts Δz satisfying $\varepsilon \ll |\Delta z| \ll \dfrac{c_{min}}{\omega_0}$.

Compute the matrix $\mathcal{K}_{ij} := |I^{c_j}\left(z(c_j) + (\Delta z)_i\right)|^2$

$\forall c_j \in [c_{min}, c_{max}]$

$\forall (\Delta z)_i \in \mathcal{S}$

Building the estimator in practice

Pick a travel time $t_0 \gg \omega_0^{-1}$.

Define the set of possible values for c : $[c_{min}, c_{max}]$. Define the probing points $z(c) = (0, ct_0) \quad \forall c \in [c_{min}, c_{max}]$.

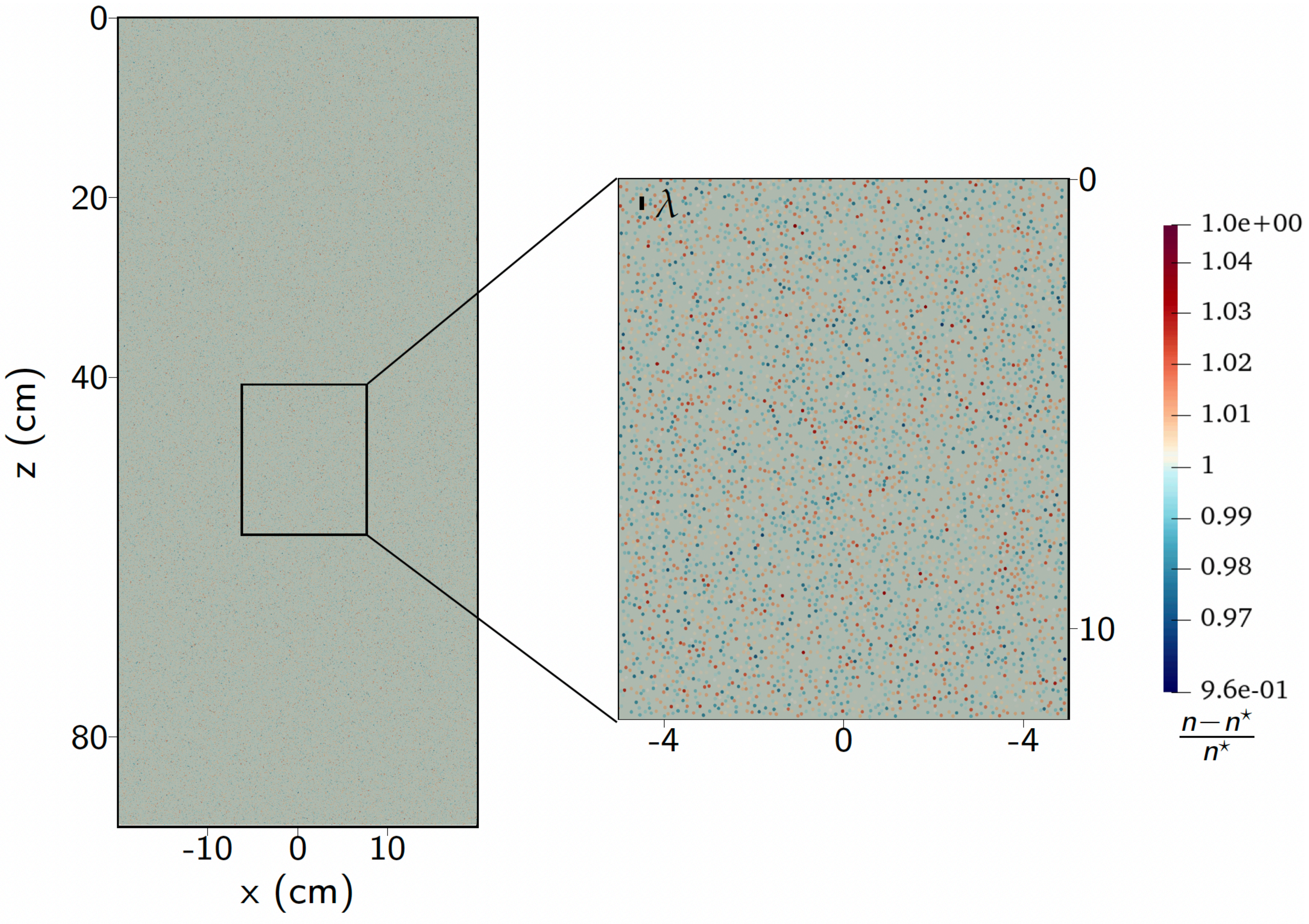
Define a set $\mathcal{S}$ of small spatial shifts Δz satisfying $\varepsilon \ll |\Delta z| \ll \dfrac{c_{min}}{\omega_0}$.

Compute the matrix $\mathcal{K}_{ij} := |I^{c_j}(z(c_j) + (\Delta z)_i)|^2$ $\quad \forall c_j \in [c_{min}, c_{max}]$, $\forall (\Delta z)_i \in \mathcal{S}$

Estimator

$$c^\star = \mathrm{argmax}_j \sum_i \mathcal{K}_{i,j}$$

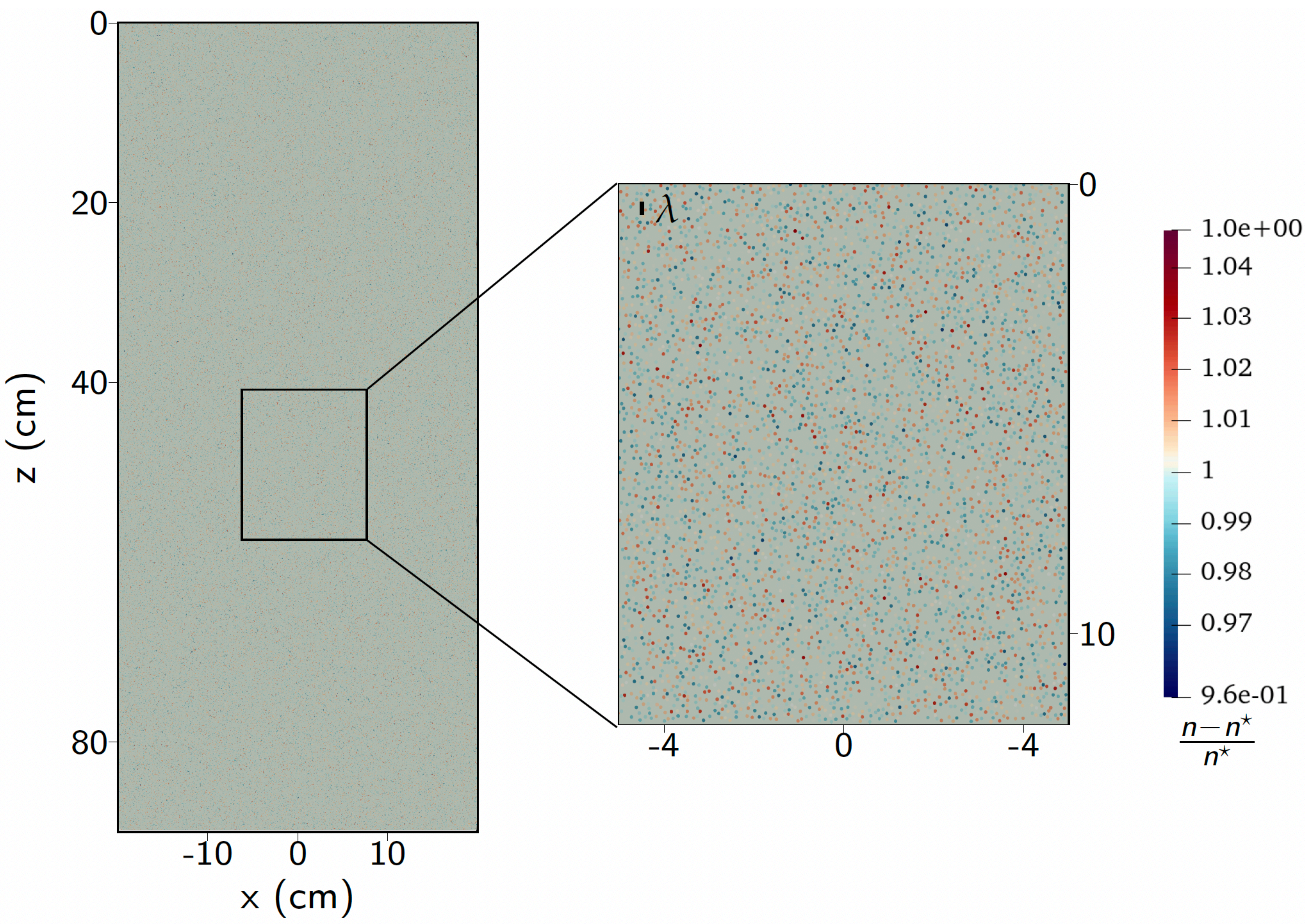
Numerical simulations (E. Parolin)



Simulations by E. **Parolin** (Alpines, INRIA)

166864 scatterers, $\varepsilon = 38.8\mu m$, $f = 4Mhz$.

Numerical simulations (E. Parolin)

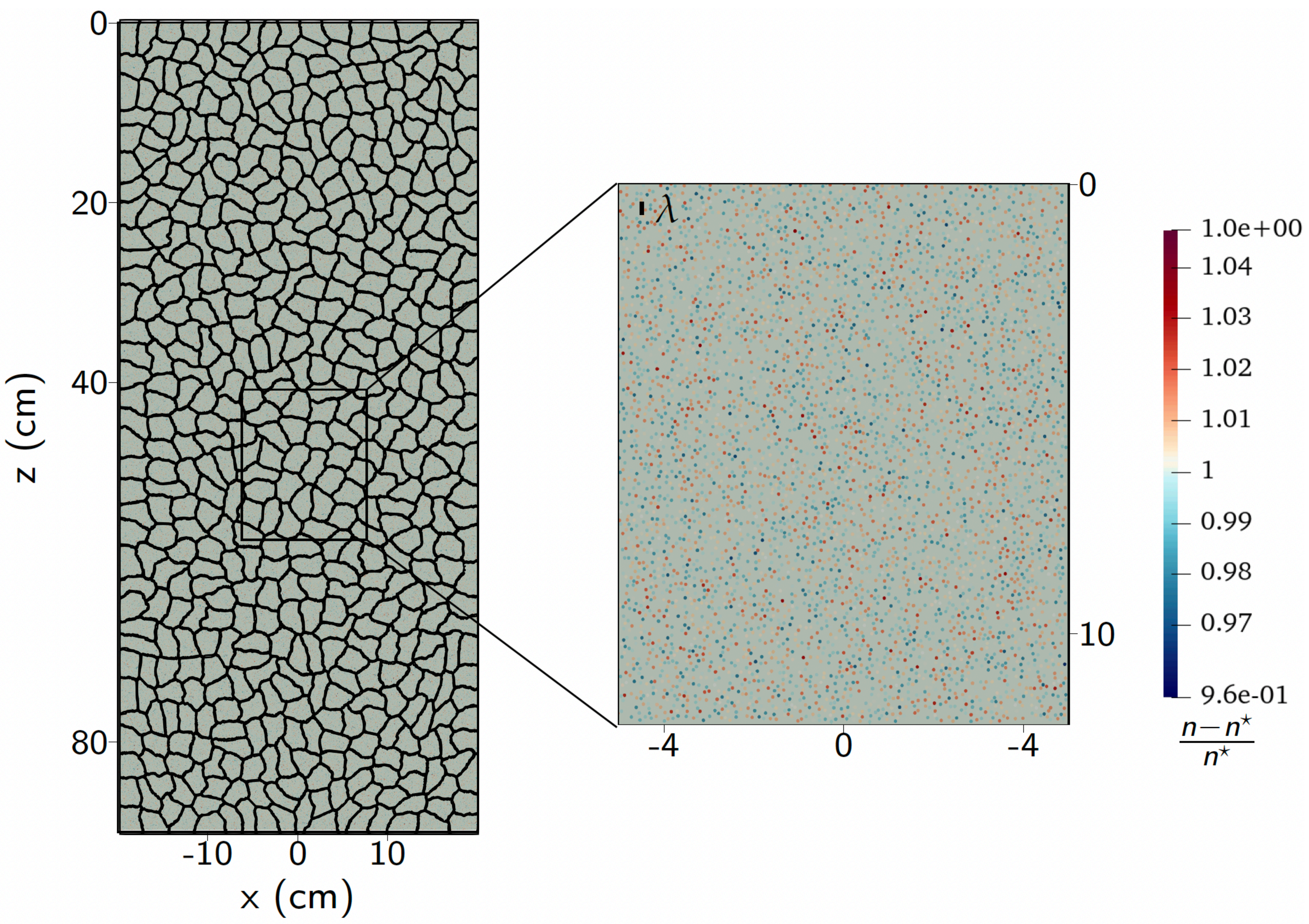


Parameters of the simulation

Discretisation: 108 DOFs (P3),
Preconditioner: one-level DD method (ORAS) with 512 subdomains,
Solver: 164 GMRES iterations for a 10–4 residual tolerance
Computation time: ~ 5 min on INRIA Paris' supercomputer.
implemented with Freefem++.

Simulations by E. **Parolin** (Alpines, INRIA)
166864 scatterers, $\varepsilon = 38.8\mu m$, $f = 4Mhz$.

Numerical simulations (E. Parolin)

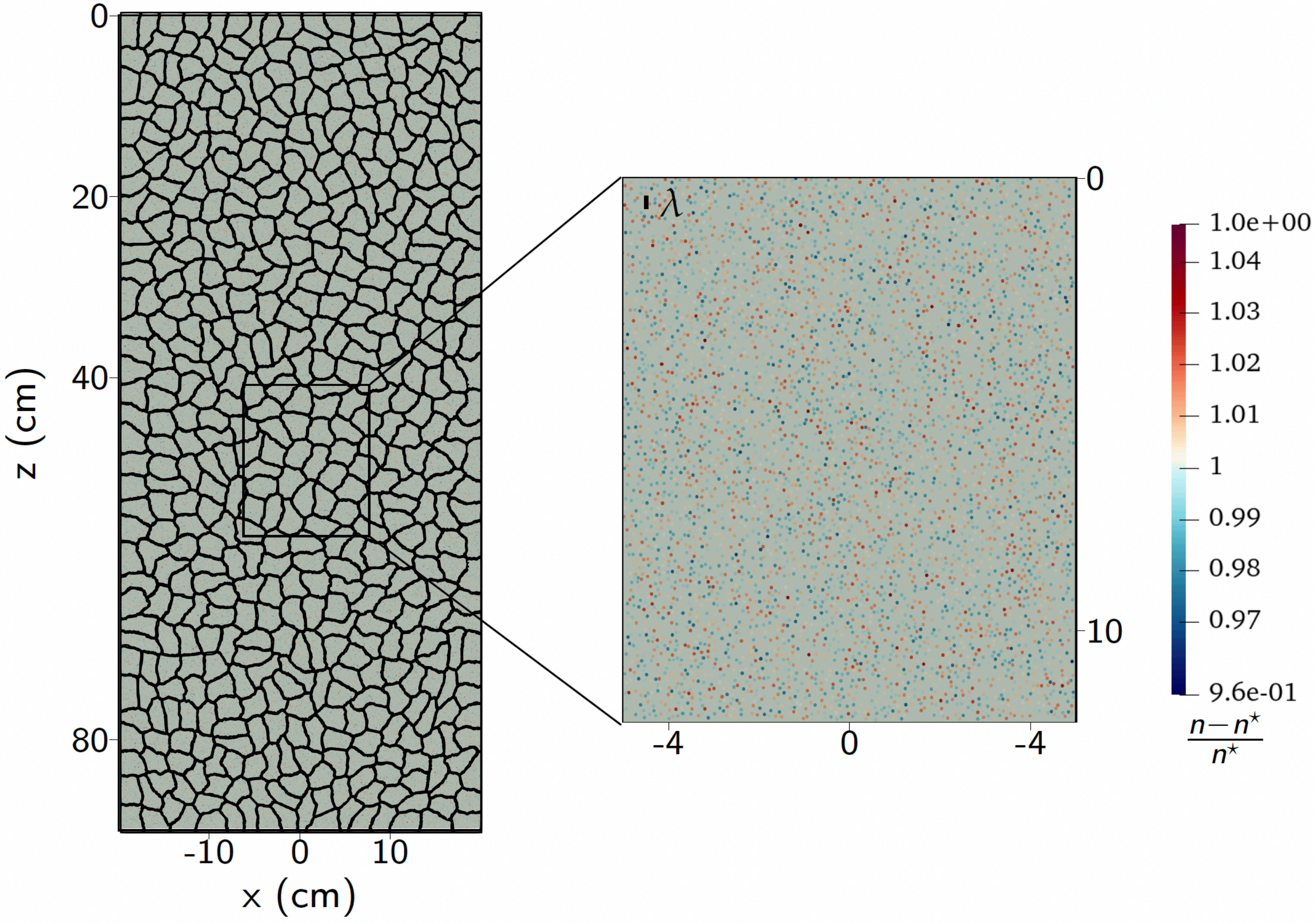


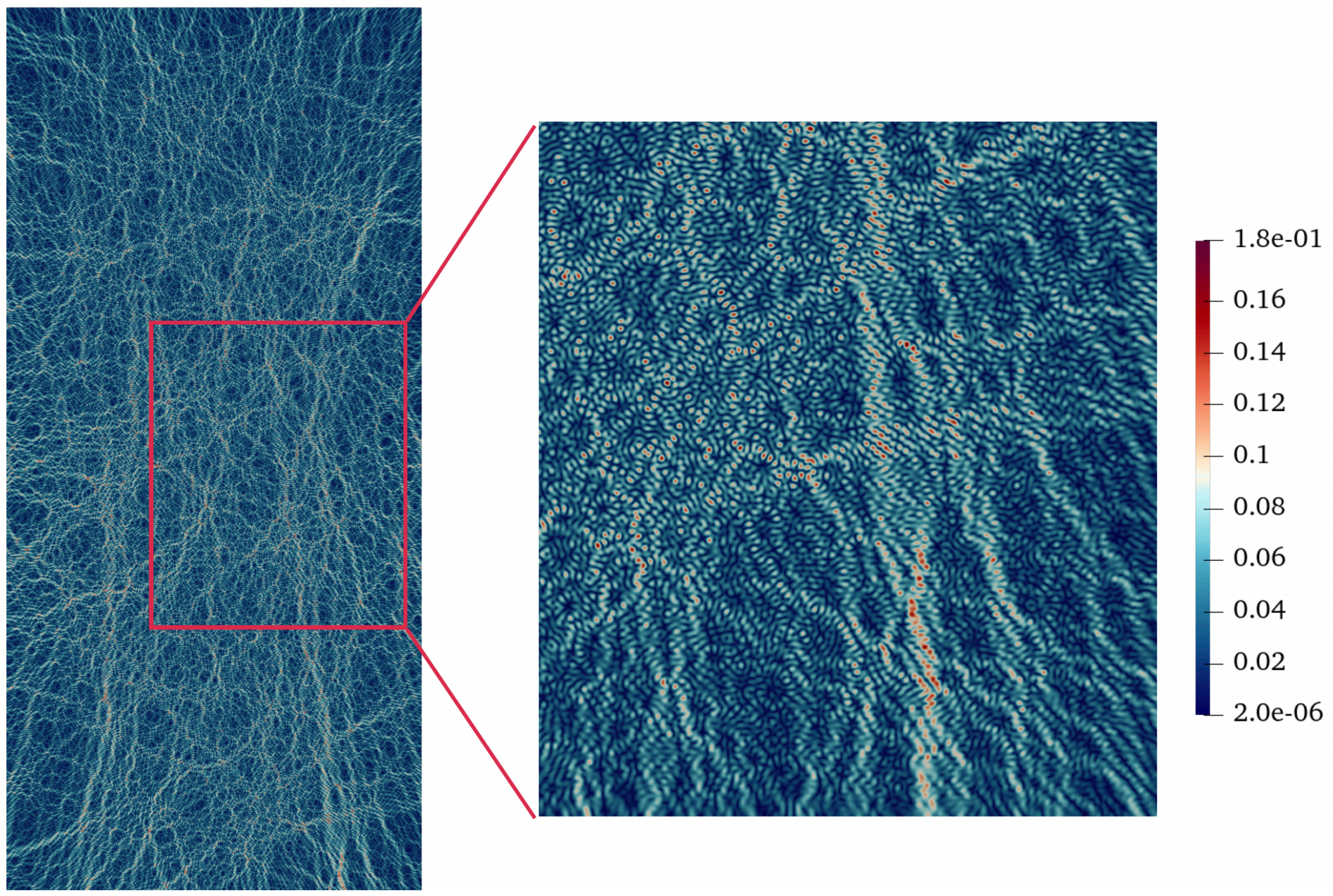
Parameters of the simulation

Discretisation: 108 DOFs (P3),
Preconditioner: one-level DD method (ORAS) with 512 subdomains,
Solver: 164 GMRES iterations for a 10–4 residual tolerance
Computation time: ~ 5 min on INRIA Paris' supercomputer.
implemented with Freefem++.

Simulations by E. **Parolin** (Alpines, INRIA)
166864 scatterers, $\varepsilon = 38.8\mu m$, $f = 4Mhz$.

Numerical simulations (E. Parolin)

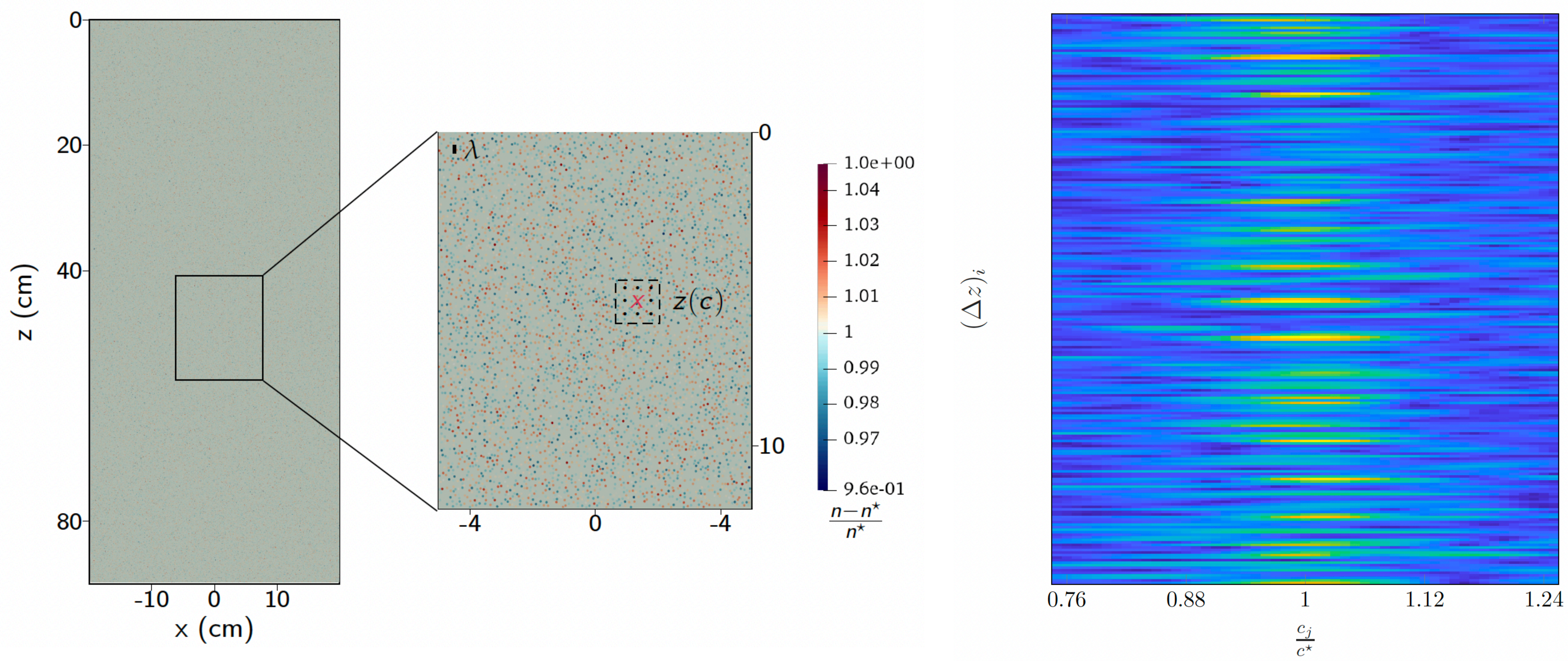




$|u^s|$

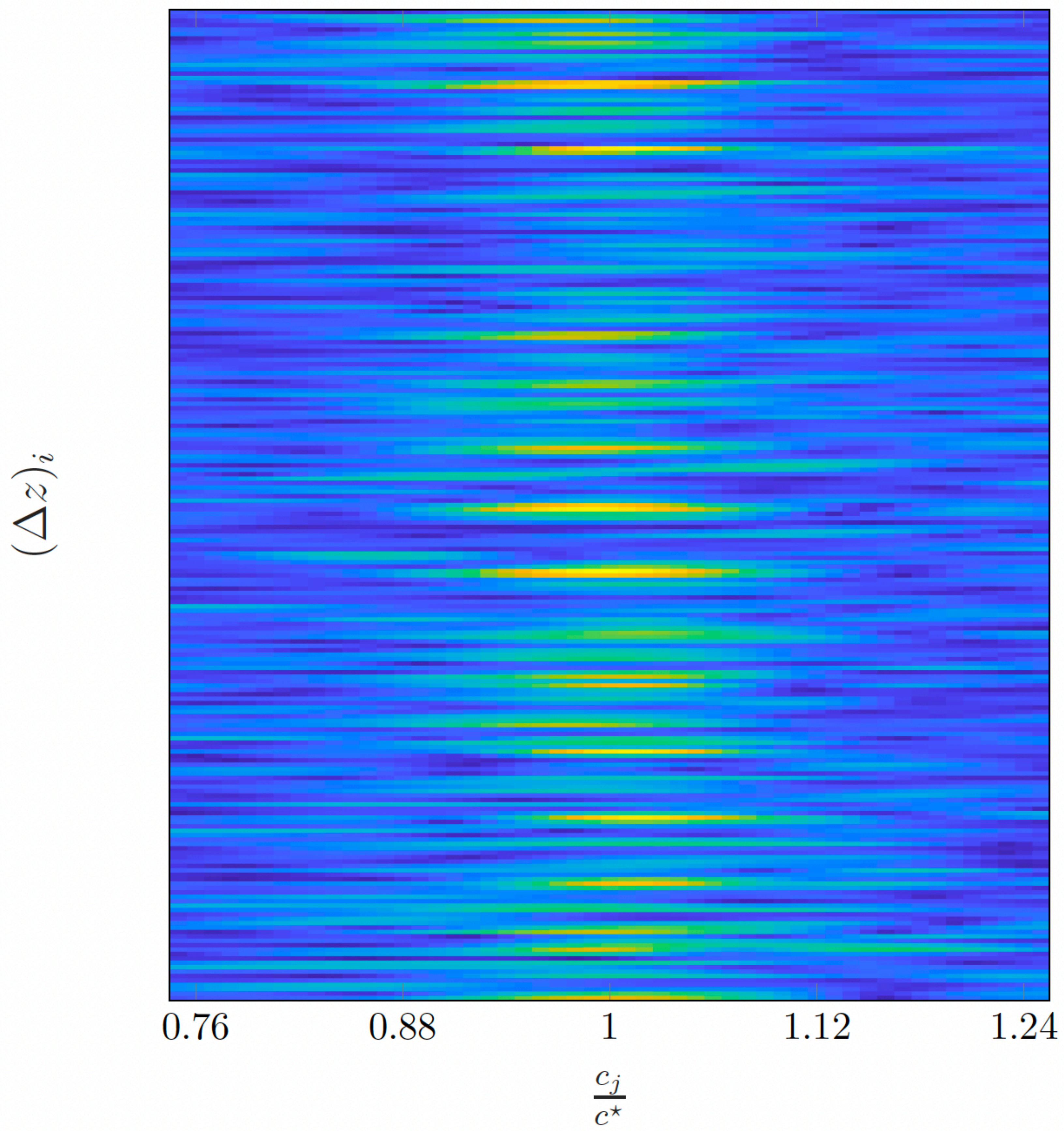
Simulations by E. **Parolin** (Alpines, INRIA)
166864 scatterers, $\varepsilon = 38.8\mu m$, $f = 4Mhz$.

Spatial averaging for a fixed realization



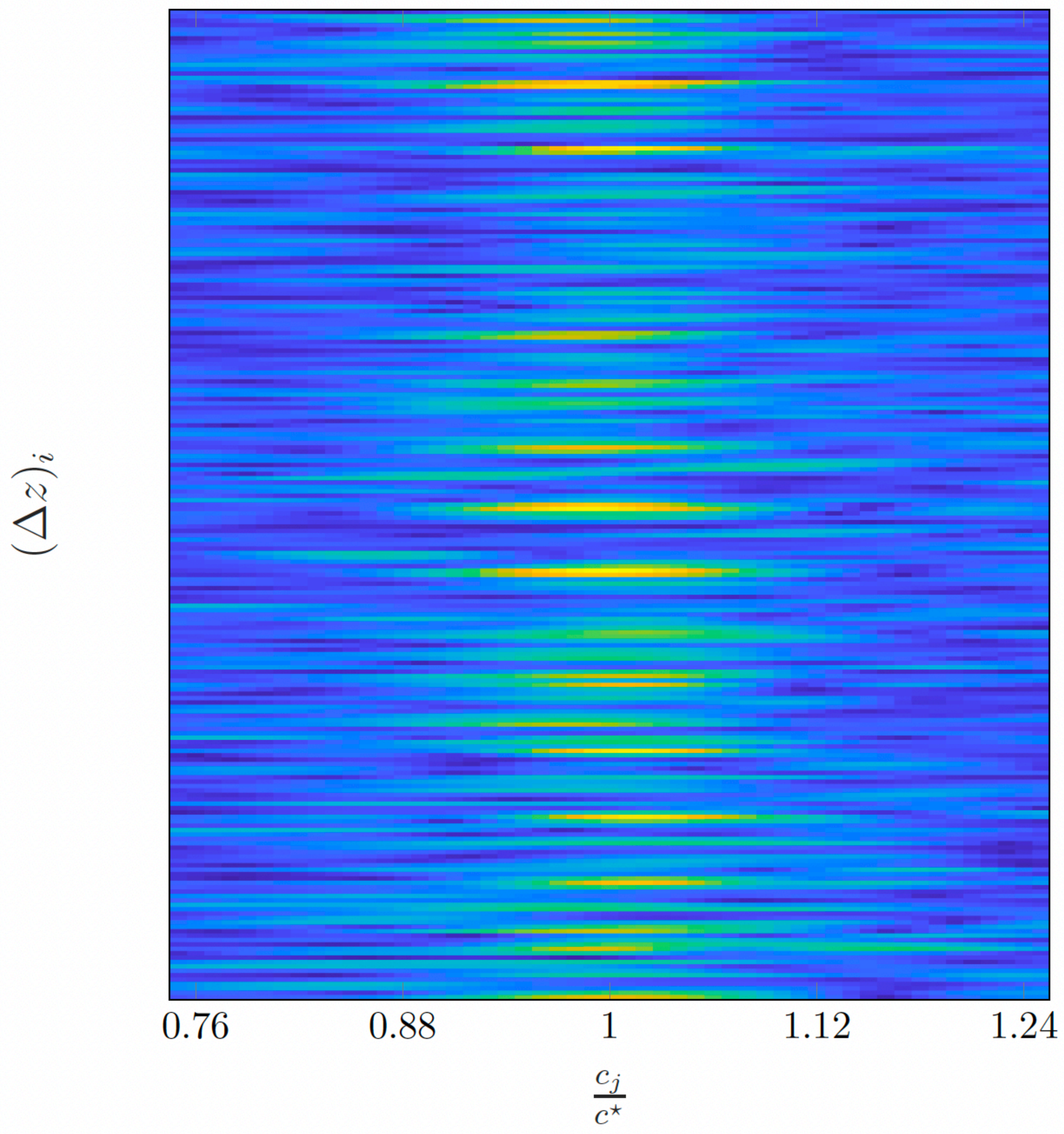
$$|\mathcal{K}_{ij}| = |I^{c_j}(z(c_j) + (\Delta z)_i, \gamma_0)|^2$$

Spatial averaging for a fixed realization

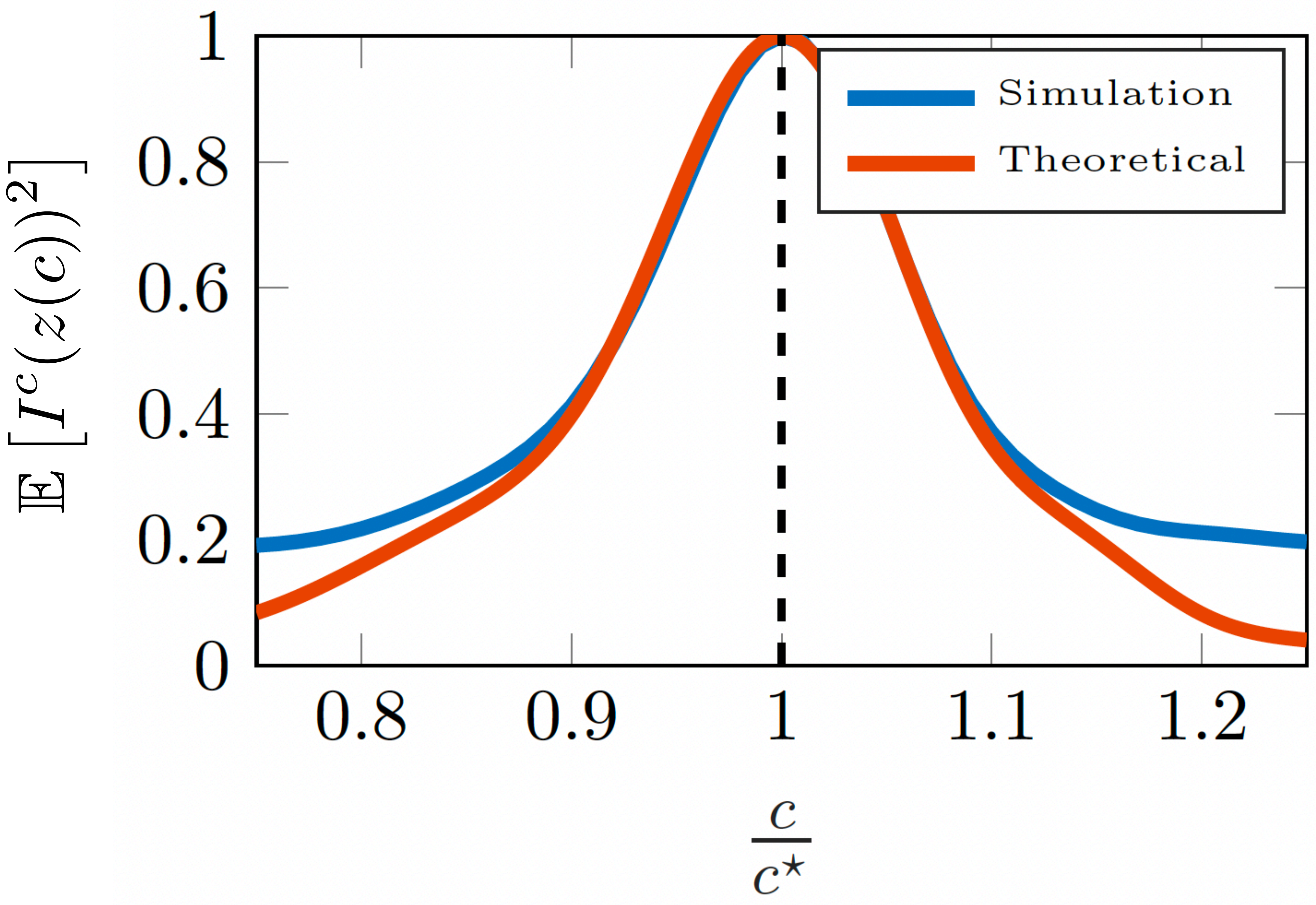


$$|\mathcal{K}_{ij}| = |I^{c_j}(z(c_j) + (\Delta z)_i, \gamma_0)|^2$$

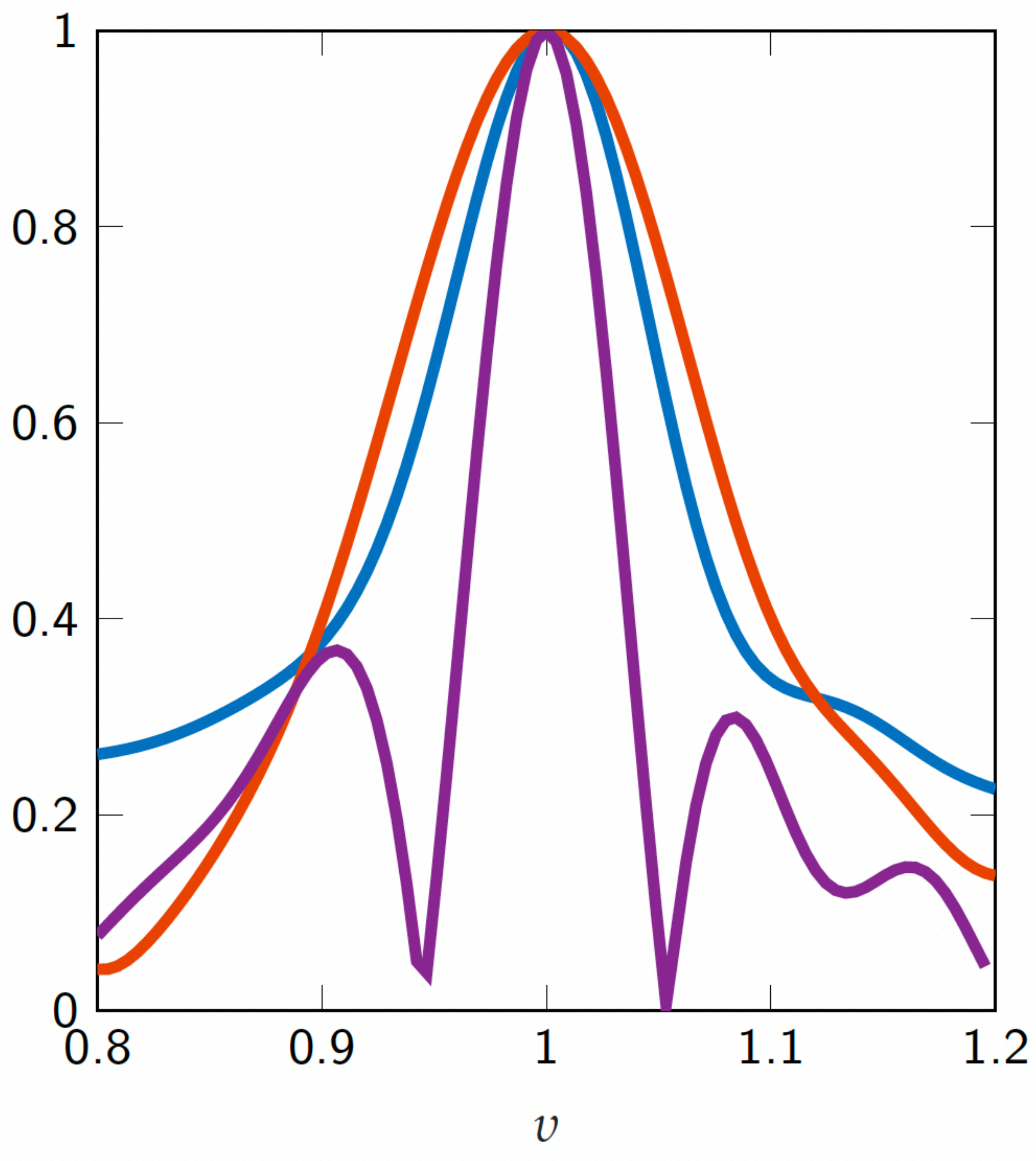
Spatial averaging for a fixed realization



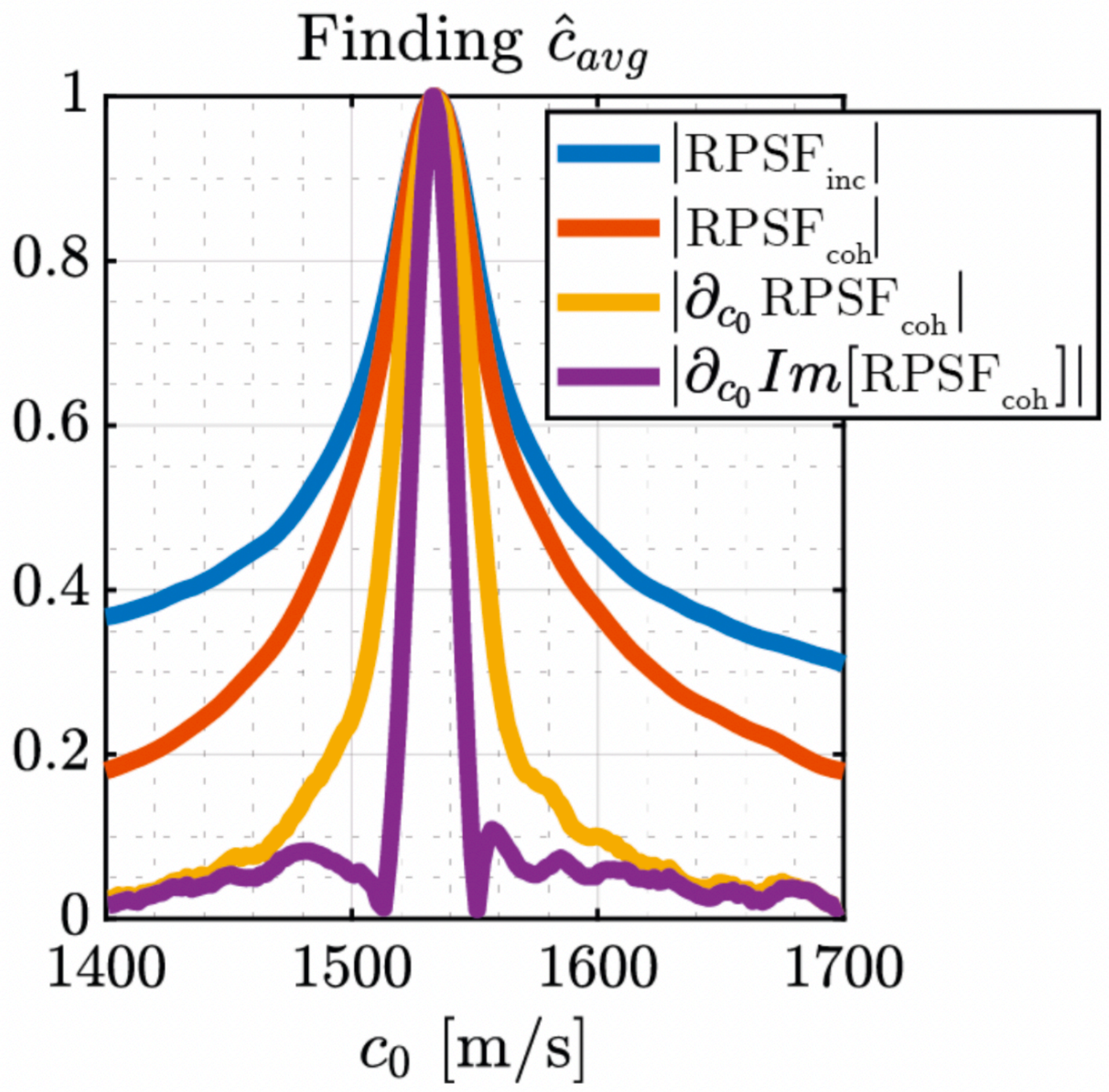
$$|\mathcal{K}_{ij}| = |I^{c_j}(z(c_j) + (\Delta z)_i, \gamma_0)|^2$$



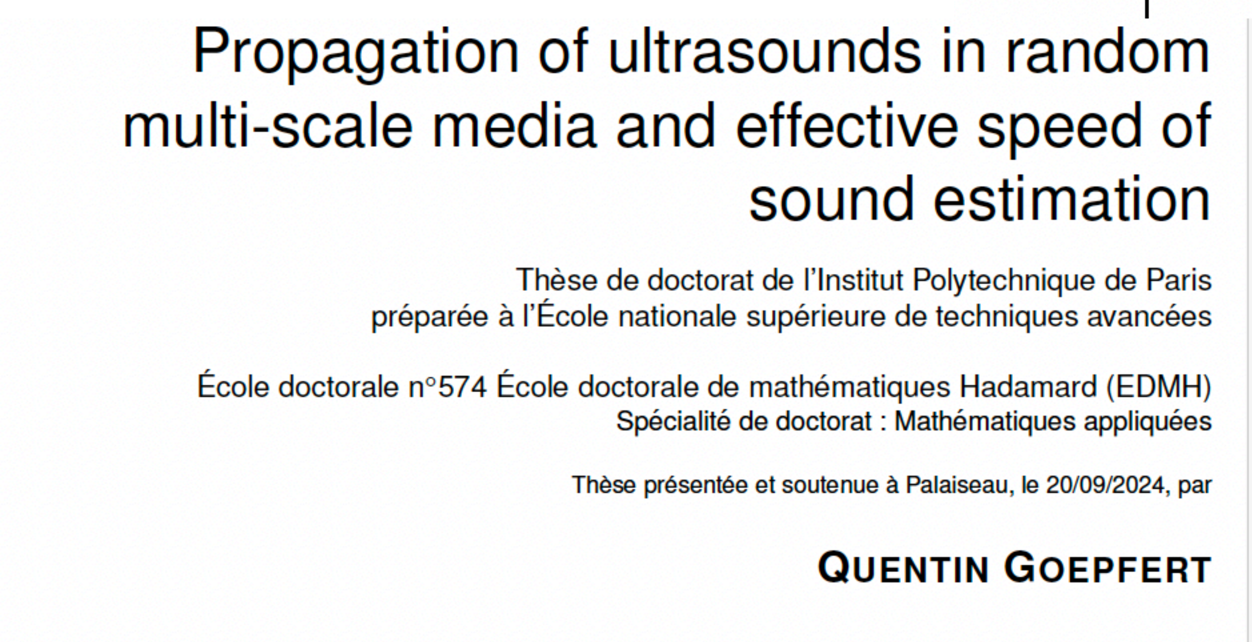
Comparison with experiments



(—) : $\hat{c_0}^{(1)}$ (—) : $\hat{c_0}^{(2)}$ (—) : $\hat{c_0}^{(3)}$.



Comparison with experiments


Propagation of ultrasounds in random multi-scale media and effective speed of sound estimation

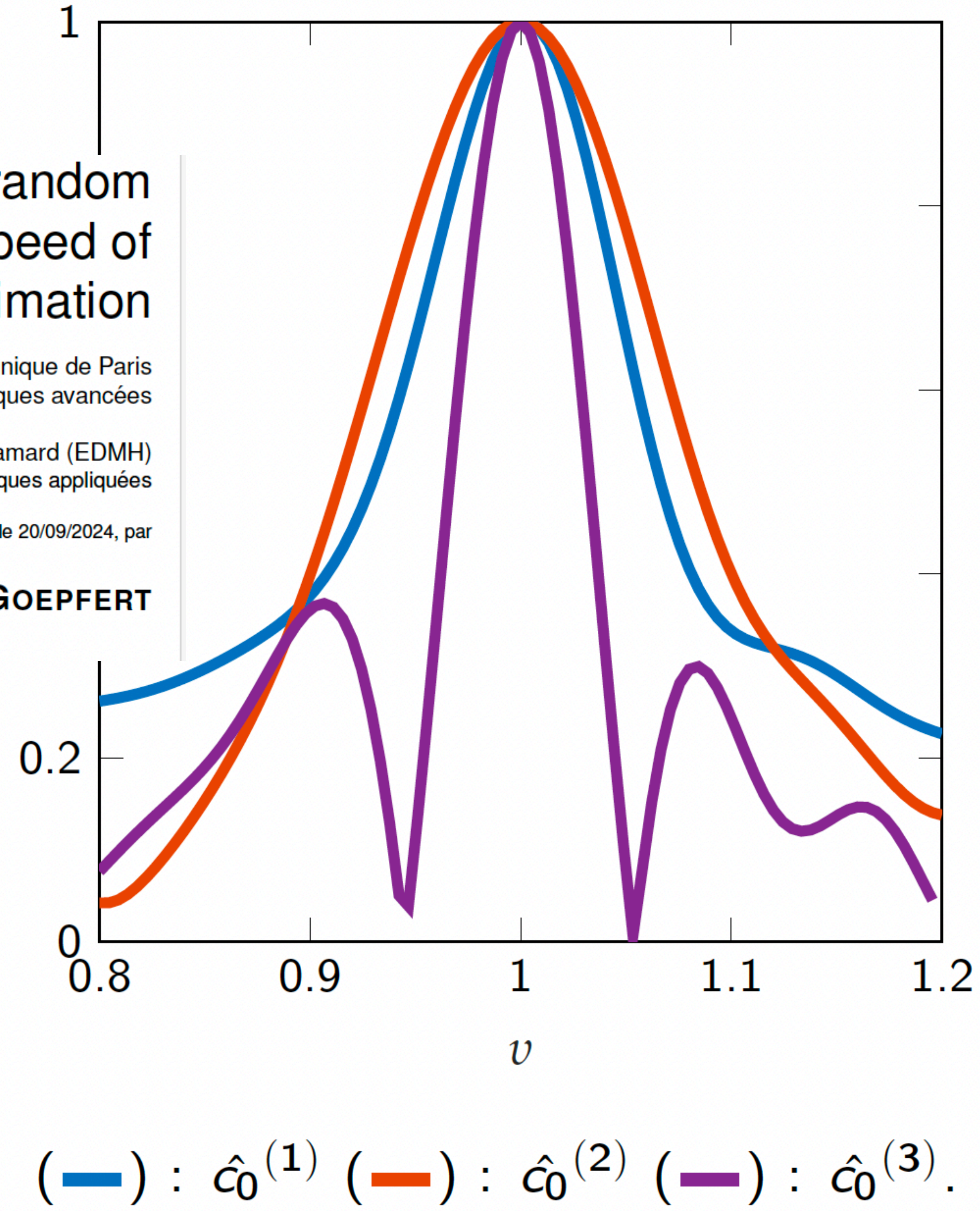
Thèse de doctorat de l'Institut Polytechnique de Paris
préparée à l'École nationale supérieure de techniques avancées

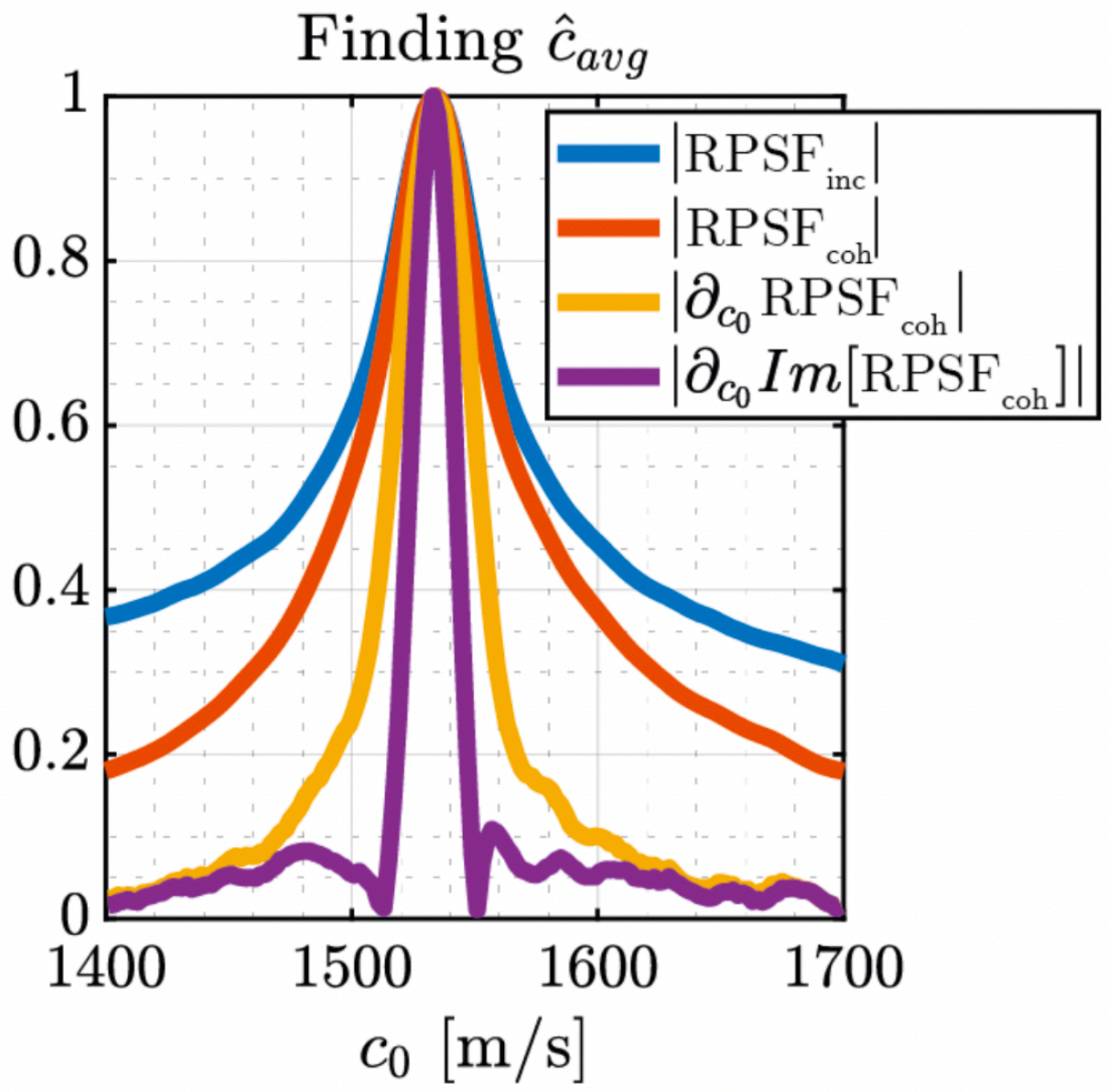
École doctorale n°574 École doctorale de mathématiques Hadamard (EDMH)
Spécialité de doctorat : Mathématiques appliquées

Thèse présentée et soutenue à Palaiseau, le 20/09/2024, par

QUENTIN GOEPPERT

Numerical simulations





Comparison with experiments

Propagation of ultrasounds in random multi-scale media and effective speed of sound estimation

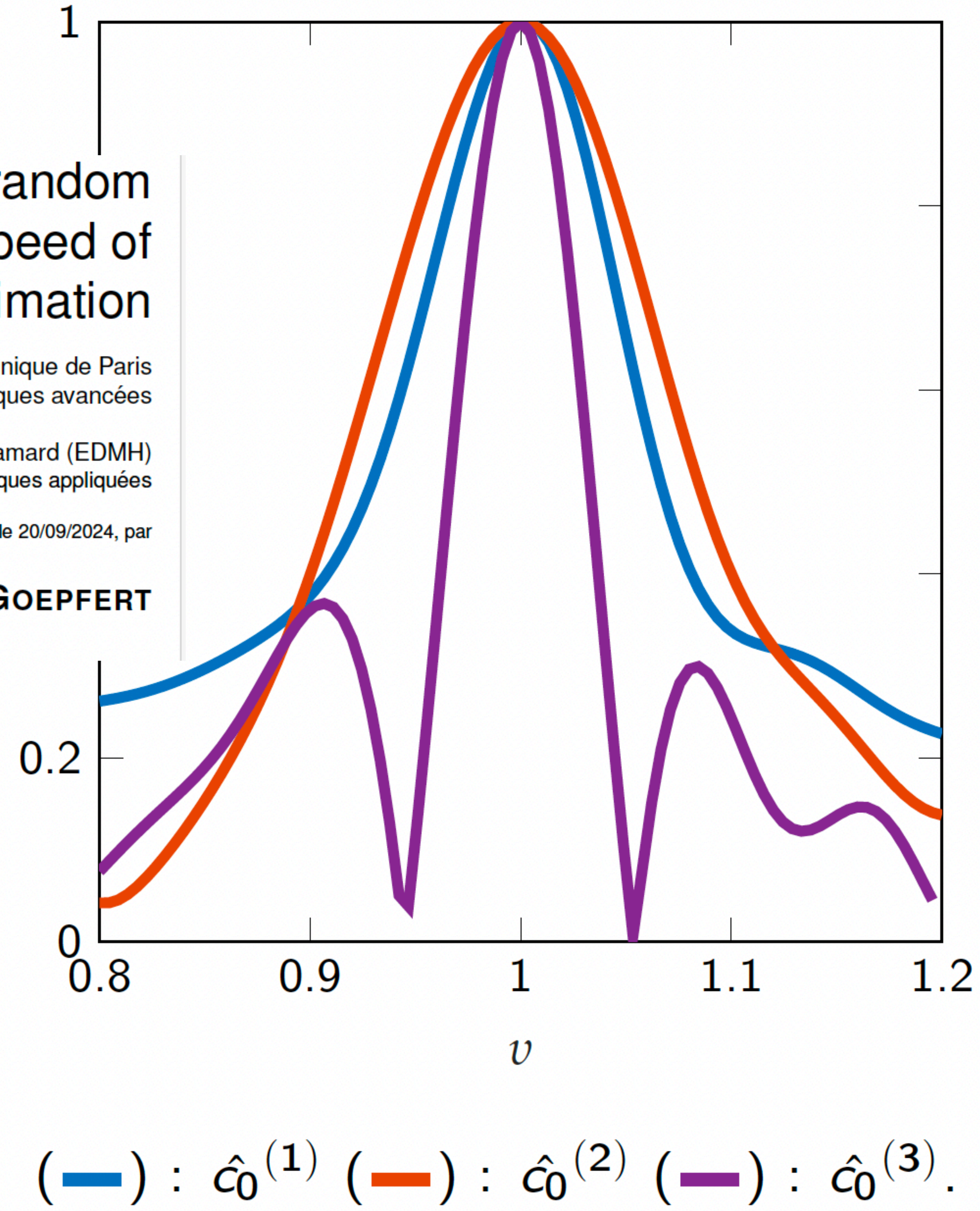
Thèse de doctorat de l'Institut Polytechnique de Paris
préparée à l'École nationale supérieure de techniques avancées

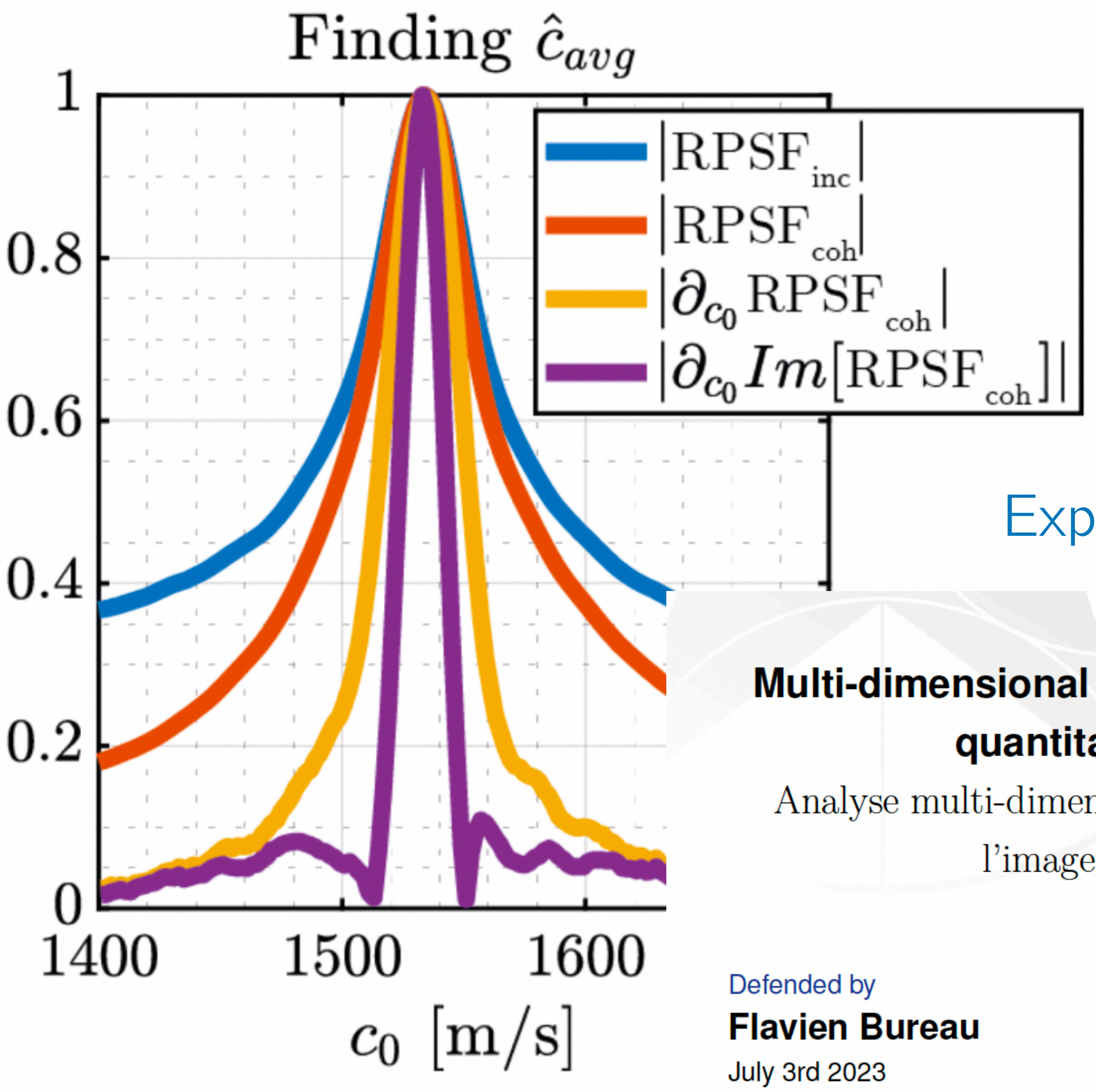
École doctorale n°574 École doctorale de mathématiques Hadamard (EDMH)
Spécialité de doctorat : Mathématiques appliquées

Thèse présentée et soutenue à Palaiseau, le 20/09/2024, par

Quentin Goeppert

Numerical simulations





Experimental curves

Multi-dimensional analysis of the reflection matrix for quantitative ultrasound imaging

Analyse multi-dimensionnelle de la matrice de réflexion pour l'imagerie ultrasonore quantitative

Defended by
Flavien Bureau
July 3rd 2023

Doctoral school n°564
Ecole Physique en Île-de-France

Speciality
Physics

Prepared at the
Institut Langevin (UMR 7587)

Composition of the jury:

Olivier Couture Research director, CNRS	*Chair of the jury* *Examiner*
Jeremy Dahl Associate Professor, Stanford Unversity	*Reviewer*
Jean Provost Associate Professor, Polytechnique Montréal	*Reviewer*
Juliette Pierre Research scientist, CNRS	*Examiner*
Mathias Fink Professor, ESPCI Paris	*Thesis supervisor*
Alexandre Aubry Research director, CNRS	*Thesis supervisor*

Summary

Modelisation

Provided a model for wave propagation (and back scattering) in a random micro-structured medium

Used this model to describe medical ultrasound imaging measurements

Inverse problem

Analysed the point-spread function when there is a mismatch modeled and actual speed of sound in the medium

Characterized the focal spot

Analysed the image quality indicator introduced by Aubry, Bureau, Fink & Al.

Proved that it is possible to access *in situ* the effective speed of sound in a micro-structured medium from backscattering measurements

Summary

Modelisation

Provided a model for wave propagation (and back scattering) in a random micro-structured medium

Used this model to describe medical ultrasound imaging measurements

Inverse problem

Analysed the point-spread function when there is a mismatch modeled and actual speed of sound in the medium

Characterized the focal spot

Analysed the image quality indicator introduced by Aubry, Bureau, Fink & Al.

Proved that it is possible to access *in situ* the effective speed of sound in a micro-structured medium from backscattering measurements

Take Home messages

It is possible to re introduce sparsity by focusing in the medium

The amplitude of the PSF is enough to control *a posteriori* the quality of the focusing (at least the geometrical component)

Perspectives

Speed of sound mapping

Extend the method to more complex speed of sound model

Perspectives

Speed of sound mapping

Extend the method to more complex speed of sound model

Layered mediums - refraction effects (ongoing work w. C. Prada)

Local variation of speed of sound (on going work w. A. Aubry's team)

Perspectives

Speed of sound mapping

Extend the method to more complex speed of sound model

Layered mediums - refraction effects (ongoing work w. C. Prada)

Local variation of speed of sound (on going work w. A. Aubry's team)

Microstructure analysis

Add some contrast in the divergence (density)

Possible effective anisotropy

Effects of multiple scattering

Extension to propagation in solids (ongoing work w. C. Prada)