Averaged Steklov eigenvalues as macroscopic indicator functions

Houssem HADDAR

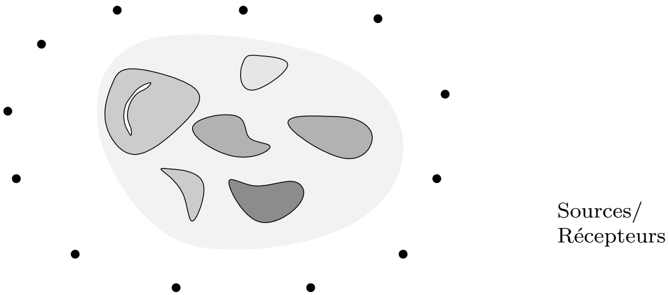
INRIA & ENSTA
Project team IDEFIX (INRIA, EDF R&D and ENSTA)

Based on a joint work with

L. Audibert and F. Pourre

WICOM 2025

Context

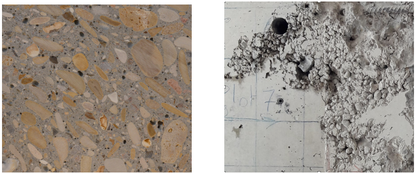


Goal: Construct "qualitative and/or quantitative" indicators on the material properties or defects inside unknown (complex) media from multi-static measurements of scattered waves.

- Forward solvers for the exact problem cannot be used since either the background cannot be correctly modeled or the defect has a complicated structure (network of cracks).

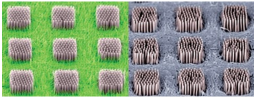
Targeted applications

- Monitor defects or material quality in concrete type materials, composite materials, etc, ...



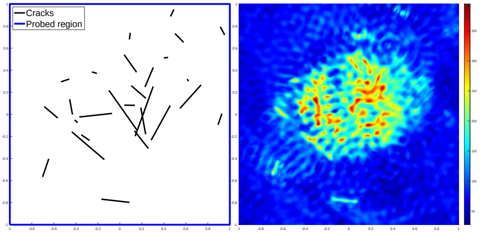
Concrete materials

- Image defects in periodic structures with unknown periodic pattern such as nano-grass structures.



Nano-grass structure

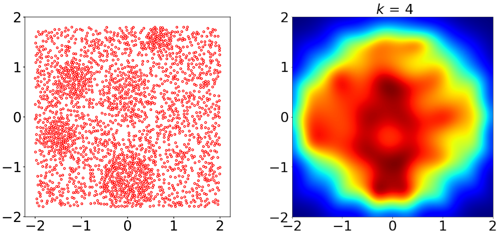
Examples of qualitative reconstructions



Images produced by “linear sampling method” using far field data [a]

[a] Other/similar non iterative imaging techniques: Liu-Sini (2010), Ammari-Garnier-Kang-Park-Sølna (2011), Bonnet (2011), Boukari-H. (2013), Bellis-Bonnet (2013), Guo-Wu-Yan (2015), Daimon-Furuya-Saiin (2020), Audibert-Chesnel-Napal-H. (2022) . . .

Examples of qualitative reconstructions



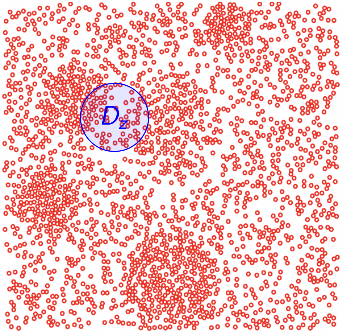
Images produced by "linear sampling method" using far field data [a]

Goal: Can we construct a "better" indicator function that also has a quantitative interpretation?

---

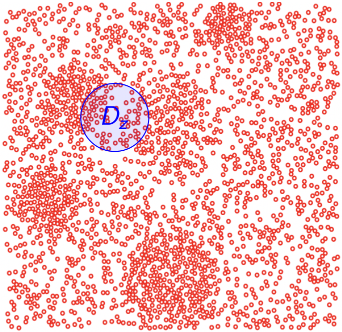
[a] Other/similar non iterative imaging techniques: Liu-Sini (2010), Ammari-Garnier-Kang-Park-Sølna (2011), Bonnet (2011), Boukari-H. (2013), Bellis-Bonnet (2013), Guo-Wu-Yan (2015), Daimon-Furuya-Saiin (2020), Audibert-Chesnel-Napal-H. (2022) ...

General scheme for the new algorithm



- At a location z, numerically introduce an artificial resonator D_z

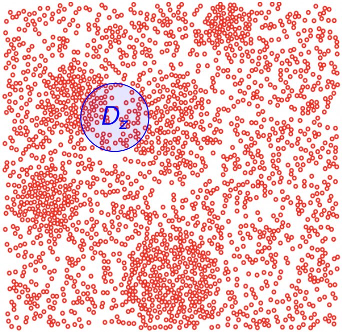
General scheme for the new algorithm



- At a location z, numerically introduce an artificial resonator D_z
- Identify the resonance parameter from measurements using the inside-outside duality :

$$\text{resonant inside} \Leftrightarrow \text{silent outside}$$

General scheme for the new algorithm

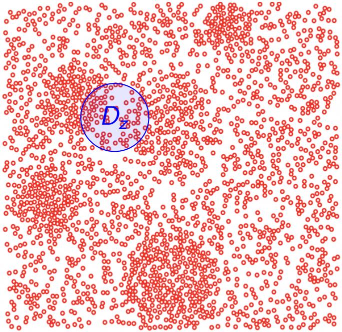


- At a location z, numerically introduce an artificial resonator D_z
- Identify the resonance parameter from measurements using the inside-outside duality :

  resonant inside $\Leftrightarrow$ silent outside

- Relate the resonance to a qualitative/quantitative indicator function $I(z)$. Compute $I(z)$ by sweeping z over inspected area

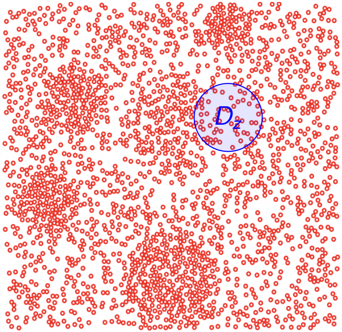
General scheme for the new algorithm



- At a location z, numerically introduce an artificial resonator D_z
- Identify the resonance parameter from measurements using the inside-outside duality :

  resonant inside $\Leftrightarrow$ silent outside
- Relate the resonance to a qualitative/quantitative indicator function $I(z)$. Compute $I(z)$ by sweeping z over inspected area

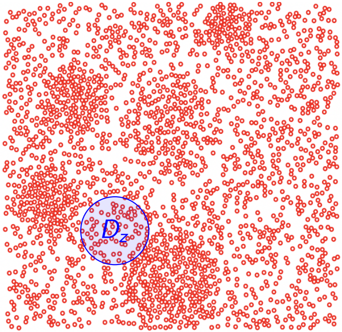
General scheme for the new algorithm



- At a location z, numerically introduce an artificial resonator D_z
- Identify the resonance parameter from measurements using the inside-outside duality :

  resonant inside $\Leftrightarrow$ silent outside
- Relate the resonance to a qualitative/quantitative indicator function $I(z)$. Compute $I(z)$ by sweeping z over inspected area

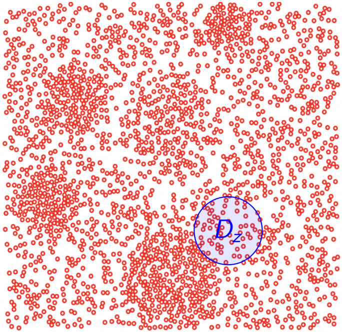
General scheme for the new algorithm



- At a location z, numerically introduce an artificial resonator D_z
- Identify the resonance parameter from measurements using the inside-outside duality :

  resonant inside $\Leftrightarrow$ silent outside
- Relate the resonance to a qualitative/quantitative indicator function $I(z)$. Compute $I(z)$ by sweeping z over inspected area

General scheme for the new algorithm



- At a location z, numerically introduce an artificial resonator D_z
- Identify the resonance parameter from measurements using the inside-outside duality :

  resonant inside $\Leftrightarrow$ silent outside
- Relate the resonance to a qualitative/quantitative indicator function $I(z)$. Compute $I(z)$ by sweeping z over inspected area

In this talk:

- Averaged-Steklov eigenvalues as resonators
- Where do they come from and why they are good candidates
- Inside - Outside duality to identify them from measurements at fixed frequency
- From far field to localized macroscopic indicator functions (link with homogenized properties)
- Generalization. From far field to a localized DtN operator.

A simple model problem

Scalar acoustic equation for inhomogeneous media

Refractive index n: $n = 1$ in $\mathbb{R}^d \setminus D$ and $\mathbb{R}^d \setminus D$ is connected.

The total field: $u \in H^1_{loc}(\mathbb{R}^d)$

$$\Delta u + k^2 n u = 0 \text{ in } \mathbb{R}^d$$

We assume that the field is generated by incident plane waves:

$$u^i(\hat{x}_0, x) := e^{ikx\cdot\hat{x}_0} \quad \hat{x}_0 \in \mathbb{S}^{d-1}$$

The scattered field

$$u^s(\hat{x}_0, \cdot) = u - u^i(\hat{x}_0, \cdot) \quad \text{in } \mathbb{R}^d,$$

satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} \int_{|x|=r} \left| \frac{\partial u^s}{\partial r} - iku^s \right|^2 ds = 0.$$

A simple model problem

Scalar acoustic equation for inhomogeneous media

Recall that with $\hat{x} := x/|x|$,

$$u^s(\hat{x}_0, x) = \frac{e^{ik|x|}}{|x|^{(d-1)/2}}\left(u^\infty(\hat{x}_0, \hat{x}) + O(1/|x|)\right)$$

Our data is formed by (noisy measurements of) so-called far field patterns

$$u^\infty(\hat{x}_0, \hat{x}) \text{ for all } (\hat{x}_0, \hat{x}) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$$

Far field Operator: $F : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$, defined by

$$Fg(\hat{x}) := \int_{\mathbb{S}^{d-1}} u^\infty(\hat{x}_0, \hat{x}) g(\hat{x}_0) ds(\hat{x}_0).$$

A simple model problem

Scalar acoustic equation for inhomogeneous media

Recall that with $\hat{x} := x/|x|$,

$$u^s(\hat{x}_0, x) = \frac{e^{ik|x|}}{|x|^{(d-1)/2}}\left(u^\infty(\hat{x}_0, \hat{x}) + O(1/|x|)\right)$$

Our data is formed by (noisy measurements of) so-called far field patterns

$$u^\infty(\hat{x}_0, \hat{x}) \text{ for all } (\hat{x}_0, \hat{x}) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$$

Far field Operator: $F : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$, defined by

$$Fg(\hat{x}) := \int_{\mathbb{S}^{d-1}} u^\infty(\hat{x}_0, \hat{x}) g(\hat{x}_0) ds(\hat{x}_0).$$

Fg is the far field generated by an incident field $u^i = v_g$ where

$$v_g(x) := \int_{\mathbb{S}^{d-1}} u^i(\hat{x}_0, x) g(\hat{x}_0) ds(\hat{x}_0), \; g \in L^2(\mathbb{S}^{d-1}), \; x \in \mathbb{R}^d.$$

Averaged-Steklov eigenvalues and Inside-Outside duality

L. Audibert, F. Cakoni, and H.H., **New sets of eigenvalues in inverse scattering for inhomogeneous media and their determination from scattering data,** (2017)

L. Audibert, H.H. and F. Pourre, **Imaging highly heterogeneous media using modified transmission eigenvalues,** (2022-2024)

Averaged-Steklov eigenvalue problem

Definition Averaged-Steklov eigenvalues for n and a domain D_b are $\mu \in \mathbb{R}$ such that there exists a non trivial solution $w \in H^1(D_b)$ such that

$$\begin{cases} \Delta w + k^2 n w = 0 \text{ in } D_b, \\ \mu w + \int_{\partial D_b} \partial_\nu w ds = 0 \text{ on } \partial D_b, \end{cases} \tag{1}$$

Averaged-Steklov eigenvalue problem

Definition Averaged-Steklov eigenvalues for n and a domain D_b are $\mu \in \mathbb{R}$ such that there exists a non trivial solution $w \in H^1(D_b)$ such that

$$\begin{cases} \Delta w + k^2 n w = 0 \text{ in } D_b, \\ \mu w + \int_{\partial D_b} \partial_\nu w ds = 0 \text{ on } \partial D_b, \end{cases} \tag{1}$$

Where does this problem come from?

Averaged-Steklov eigenvalue problem

Definition Averaged-Steklov eigenvalues for n and a domain D_b are $\mu \in \mathbb{R}$ such that there exists a non trivial solution $w \in H^1(D_b)$ such that

$$\begin{cases} \Delta w + k^2 n w = 0 \text{ in } D_b, \\ \mu w + \int_{\partial D_b} \partial_\nu w ds = 0 \text{ on } \partial D_b, \end{cases} \quad (1)$$

Where does this problem come from?

$$\begin{cases} \Delta u + k^2 n u = 0 & \text{in } D_b \\ -a\Delta v + k^2 \lambda v = 0 & \text{in } D_b \\ u = v & \text{on } \partial D_b \\ \dfrac{\partial u}{\partial \nu} = -a\dfrac{\partial v}{\partial \nu} & \text{on } \partial D_b \end{cases}$$

The largest eigenvalue $\lambda \longrightarrow \frac{\mu}{k^2|D_b|}$ as $a \to +\infty$

Averaged-Steklov eigenvalue problem

Definition Averaged-Steklov eigenvalues for n and a domain D_b are $\mu \in \mathbb{R}$ such that there exists a non trivial solution $w \in H^1(D_b)$ such that

$$\begin{cases} \Delta w + k^2 n w = 0 \text{ in } D_b, \\ \mu w + \int_{\partial D_b} \partial_\nu w ds = 0 \text{ on } \partial D_b, \end{cases} \tag{1}$$

Theorem 1: Except for a discrete set of values for k^2, problem (1) has a unique (generically) non trivial eigenvalue $\mu_1(n, D_b)$.

$$\mu_1(n, D_b) = -\int_{\partial D_b} \partial_\nu w_1 ds = k^2 \int_{D_b} n w_1 dx$$

where $w_1 \in H^1(D_b)$ is the unique solution of

$$\begin{cases} \Delta w_1 + k^2 n w_1 = 0 \quad \text{in } D_b \\ w_1 = 1 \quad \text{on } \partial D_b. \end{cases}$$

Averaged-Steklov eigenvalue problem

Definition Averaged-Steklov eigenvalues for n and a domain D_b are $\mu \in \mathbb{R}$ such that there exists a non trivial solution $w \in H^1(D_b)$ such that

$$\begin{cases} \Delta w + k^2 n w = 0 \text{ in } D_b, \\ \mu w + \int_{\partial D_b} \partial_\nu w ds = 0 \text{ on } \partial D_b, \end{cases} \tag{1}$$

Theorem 2: Denote by $\Lambda_0(n, D_b)$ the first eigenvalue of

$$\begin{cases} -\Delta w = \Lambda\, n w \quad \text{in } D_b \\ w = 0 \quad \text{on } \partial D_b \end{cases}$$

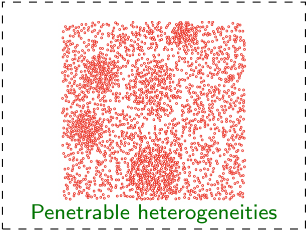
Then $\mu_1(n, D_b) > 0$ and $\mu_1(n, D_b) \to +\infty$ as $k^2 \to \Lambda_0(n, D_b)$.

Moreover we have the monotonicity property (for real n)

$$\boxed{n_1 \leq n_2 \Longrightarrow \mu_1(n_1, D_b) \leq \mu_1(n_2, D_b)}$$

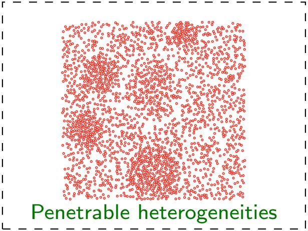
if $k^2 < \Lambda_0(n_i, D_b)$, $i = 1, 2$.

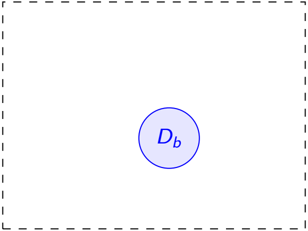
Dual definition



$$\begin{cases} \Delta u + k^2 n u = 0 \quad \text{in } \mathbb{R}^d \\ n \neq 1 \text{ in } D \text{ and } n = 1 \text{ in } \mathbb{R}^d \setminus D \\ u = u^s + u^i \text{ in } \mathbb{R}^d \\ \lim_{r=|x| \to \infty} r \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0 \end{cases}$$

Dual definition

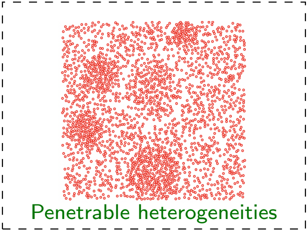


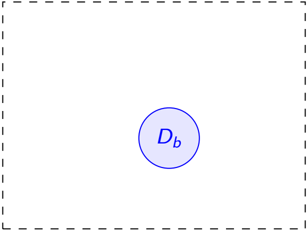


$$\begin{cases} \Delta u + k^2 n u = 0 \quad \text{in } \mathbb{R}^d \\ n \neq 1 \text{ in } D \text{ and } n = 1 \text{ in } \mathbb{R}^d \setminus D \\ u = u^s + u^i \text{ in } \mathbb{R}^d \\ \lim_{r=|x|\to\infty} r \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0 \end{cases}$$

$$\begin{cases} \Delta u_b + k^2 u_b = 0 \quad \text{in } \mathbb{R}^d \setminus D_b \\ \mu u_b + \int_{\partial D_b} \partial_\nu u_b ds = 0 \text{ on } \partial D_b, \\ u_b = u_b^s + u^i \text{ in } \mathbb{R}^d \setminus \overline{D_b} \\ \lim_{r=|x|\to\infty} r \left(\frac{\partial u_b^s}{\partial r} - iku_b^s \right) = 0 \end{cases}$$

Dual definition





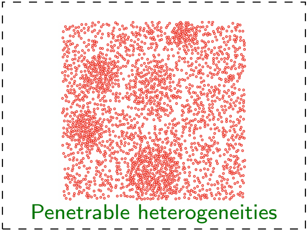
$$\begin{cases} \Delta u + k^2 n u = 0 \quad \text{in } \mathbb{R}^d \\ n \neq 1 \text{ in } D \text{ and } n = 1 \text{ in } \mathbb{R}^d \setminus D \\ u = u^s + u^i \text{ in } \mathbb{R}^d \\ \lim_{r=|x|\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \end{cases}$$

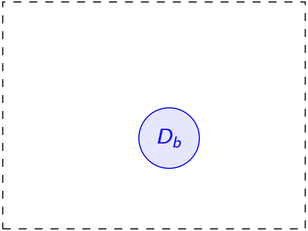
$$\begin{cases} \Delta u_b + k^2 u_b = 0 \quad \text{in } \mathbb{R}^d \setminus D_b \\ \mu u_b + \int_{\partial D_b} \partial_\nu u_b ds = 0 \text{ on } \partial D_b, \\ u_b = u_b^s + u^i \text{ in } \mathbb{R}^d \setminus \overline{D_b} \\ \lim_{r=|x|\to\infty} r\left(\frac{\partial u_b^s}{\partial r} - iku_b^s\right) = 0 \end{cases}$$

$\mu = \mu_1(n, D_b)$ if and only if there exists a non trivial incident field $u^i = v \in H^1(D_b) \cup L^2(D)$ such that

$$u_b^s \equiv u^s \text{ in } \mathbb{R}^d \setminus \{D_b \cup D\}.$$

Dual definition





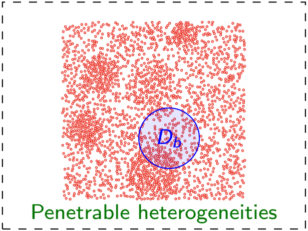
$$\begin{cases} \Delta u + k^2 n u = 0 \quad \text{in } \mathbb{R}^d \\ n \neq 1 \text{ in } D \text{ and } n = 1 \text{ in } \mathbb{R}^d \setminus D \\ u = u^s + u^i \text{ in } \mathbb{R}^d \\ \lim_{r=|x|\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \end{cases}$$

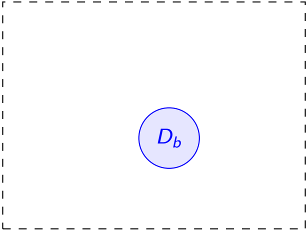
$$\begin{cases} \Delta u_b + k^2 u_b = 0 \quad \text{in } \mathbb{R}^d \setminus D_b \\ \mu u_b + \int_{\partial D_b} \partial_\nu u_b ds = 0 \text{ on } \partial D_b, \\ u_b = u_b^s + u^i \text{ in } \mathbb{R}^d \setminus \overline{D_b} \\ \lim_{r=|x|\to\infty} r\left(\frac{\partial u_b^s}{\partial r} - iku_b^s\right) = 0 \end{cases}$$

$\mu = \mu_1(n, D_b)$ if and only if there exists a sequence of incident fields (u_ϵ^i) converging to some non trivial $v \in H^1(D_b) \cup L^2(D)$ such that

$$u_b^s(v) \equiv u^s(v) \text{ in } \mathbb{R}^d \setminus \{D_b \cup D\}.$$

Dual definition





$$\begin{cases} \Delta u + k^2 n u = 0 \quad \text{in } \mathbb{R}^d \\ n \neq 1 \text{ in } D \text{ and } n = 1 \text{ in } \mathbb{R}^d \setminus D \\ u = u^s + u^i \text{ in } \mathbb{R}^d \\ \lim_{r=|x|\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \end{cases}$$

$$\begin{cases} \Delta u_b + k^2 u_b = 0 \quad \text{in } \mathbb{R}^d \setminus D_b \\ \mu u_b + \int_{\partial D_b} \partial_\nu u_b ds = 0 \text{ on } \partial D_b, \\ u_b = u_b^s + u^i \text{ in } \mathbb{R}^d \setminus \overline{D_b} \\ \lim_{r=|x|\to\infty} r\left(\frac{\partial u_b^s}{\partial r} - iku_b^s\right) = 0 \end{cases}$$

$\mu = \mu_1(n, D_b)$ if and only if there exists a sequence of incident fields (u_ϵ^i) converging to some non trivial $v \in H^1(D_b) \cup L^2(D)$ such that

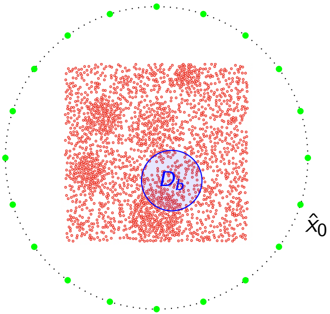
$$u_b^s(v) \equiv u^s(v) \text{ in } \mathbb{R}^d \setminus \{D_b \cup D\}.$$

Dual definition with far field data

$$u^i = u^i(\hat{x}_0, x) := e^{ikx\cdot\hat{x}_0} \quad \hat{x}_0 \in \mathbb{S}^{d-1}$$

$$u^s(\hat{x}_0, \cdot) \longrightarrow u^\infty(\hat{x}_0, \cdot)$$

$$u_b^{\mu,s}(\hat{x}_0, \cdot) \longrightarrow u_b^{\mu,\infty}(\hat{x}_0, \cdot)$$

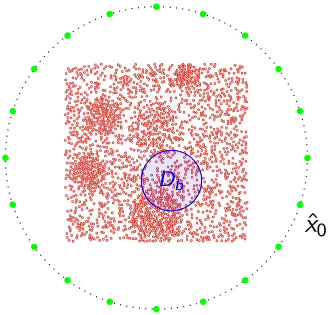


Dual definition with far field data

$u^i = u^i(\hat{x}_0, x) := e^{ikx\cdot\hat{x}_0} \quad \hat{x}_0 \in \mathbb{S}^{d-1}$

$u^s(\hat{x}_0, \cdot) \longrightarrow u^\infty(\hat{x}_0, \cdot)$

$u_b^{\mu,s}(\hat{x}_0, \cdot) \longrightarrow u_b^{\mu,\infty}(\hat{x}_0, \cdot)$



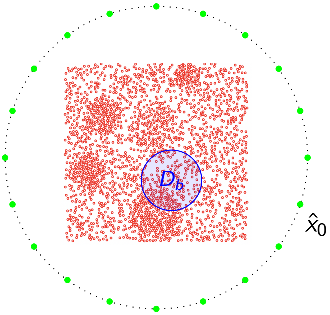
$$\boxed{Fg := \int_{\mathbb{S}^{d-1}} u^\infty(\hat{x}_0, \cdot) g(\hat{x}_0) ds(\hat{x}_0)} \text{ and } \boxed{F_b^\mu g := \int_{\mathbb{S}^{d-1}} u_b^{\mu,\infty}(\hat{x}_0, \cdot) g(\hat{x}_0) ds(\hat{x}_0)}$$

Dual definition with far field data

$$u^i = u^i(\hat{x}_0, x) := e^{ikx\cdot\hat{x}_0} \quad \hat{x}_0 \in \mathbb{S}^{d-1}$$

$$u^s(\hat{x}_0, \cdot) \longrightarrow u^\infty(\hat{x}_0, \cdot)$$

$$u_b^{\mu,s}(\hat{x}_0, \cdot) \longrightarrow u_b^{\mu,\infty}(\hat{x}_0, \cdot)$$



$$Fg := \int_{\mathbb{S}^{d-1}} u^\infty(\hat{x}_0, \cdot) g(\hat{x}_0) ds(\hat{x}_0) \quad \text{and} \quad F_b^\mu g := \int_{\mathbb{S}^{d-1}} u_b^{\mu,\infty}(\hat{x}_0, \cdot) g(\hat{x}_0) ds(\hat{x}_0)$$

Theorem: $\mu = \mu_1(n, D_b)$ if and only if there exists a sequence $(g_\epsilon) \in L^2(\mathbb{S}^{d-1})$ such that v_{g_ϵ} converges to some non trivial $v \in H^1(D_b) \cup L^2(D)$ and

$$\|Fg_\epsilon - F_b^\mu g_\epsilon\|_{L^2(\mathbb{S}^{d-1}} \to 0$$

Determination of TEs from far field data

Difficulty: $F - F_b^{\mu}$ is a compact operator... Impossible to numerically test the non injectivity of this operator.

Determination of TEs from far field data

Difficulty: $F - F_b^\mu$ is a compact operator... Impossible to numerically test the non injectivity of this operator.

Two methods can be used to overcome this difficulty.

- GLSM method:[a] Transform the non-injectivity into the non solvability of the far field equation used by the Linear Sampling Method.
  - Works for limited aperture
  - Needs a careful tuning of a regularization parameter
  - Hard to exploit for imaging if the eigenvalue is not isolated

[a] Cakoni-Colton-H. (2011), Audibert-H. (2014), Audibert-Cakoni-H. (2018), Audibert-H.-Pourre (2023), ...

Determination of TEs from far field data

Difficulty: $F - F_b^\mu$ is a compact operator... Impossible to numerically test the non injectivity of this operator.

Two methods can be used to overcome this difficulty.

- Inside-Outside duality:[a] Use the phase of the eigenvalues of $F - F_b^\mu$ that accumulate at 0 as μ approaches the averaged Steklov eigenvalue.
  - Numerically cheaper,
  - Requires symmetry between emitters/receivers,
  - Is not fully justified for penetrable inclusions.
  - Hard to implement if the eigenvalue is not isolated

[a] Eckmann-Pillet (1995), Kirsch-Lechleiter (2013), Lechleiter-Peters (2015), Audibert-Chesnel-H. (2019), H.-Khenissi-Mansouri (2022),...)

The inside-outside duality method for A.S.E.

Assume that the refractive index n and μ are real valued.

- The compact operators F and F_b^{μ} are normal
- The scattering operators $S := \frac{2ik}{\gamma} I + F$ and $S_b^{\mu} := \frac{2ik}{\gamma} I + F_b^{\mu}$ are unitary.[a]

[a] $\gamma = 4\pi$ for $d = 3$ and $\gamma = \sqrt{8\pi k} e^{i\frac{\pi}{4}}$ for $d = 2$

The inside-outside duality method for A.S.E.

Assume that the refractive index n and μ are real valued.

- The compact operators F and F_b^μ are normal
- The scattering operators $S := \frac{2ik}{\gamma} I + F$ and $S_b^\mu := \frac{2ik}{\gamma} I + F_b^\mu$ are unitary.[a]

Define

$$\mathbf{F}^\mu := \overline{\gamma}(S_b^\mu)^*(F - F_b^\mu),$$

- The compact operator $\mathbf{F}^\mu$ is also normal and the associated scattering operator $\mathbf{S}^\mu := \frac{2ik}{|\gamma|^2} I + \mathbf{F}^\mu$ is unitary.
- Moreover

$$\mathbf{F}^\mu = \mathcal{H}_\mu^* T_\mu \mathcal{H}_\mu.$$

with T_μ a (generically) coercive operator.

---

[a] $\gamma = 4\pi$ for $d = 3$ and $\gamma = \sqrt{8\pi k} e^{i\frac{\pi}{4}}$ for $d = 2$

The inside-outside duality method for A.S.E.

$$\mathbf{F}^{\mu} := \overline{\gamma}(S_b^{\mu})^*(F - F_b^{\mu}) = \mathcal{H}_{\mu}^* T_{\mu} \mathcal{H}_{\mu}.$$

Theorem: Assume in addition that $n - 1 \geq 0$ and $\mu > 0$ and is not a A.S.E, then

$$T_{\mu} = T_0 + K_{\mu}$$

with T_0 positive definite and K_{μ} compact.

The inside-outside duality method for A.S.E.

$$\mathbf{F}^{\mu} := \overline{\gamma}(S_b^{\mu})^*(F - F_b^{\mu}) = \mathcal{H}_{\mu}^* T_{\mu} \mathcal{H}_{\mu}.$$

Theorem: Assume in addition that $n - 1 \geq 0$ and $\mu > 0$ and is not a A.S.E, then

$$T_{\mu} = T_0 + K_{\mu}$$

with T_0 positive definite and K_{μ} compact. Therefore the eigenvalues $\mathbf{F}^{\mu}$ accumulate at 0 with positive real parts:

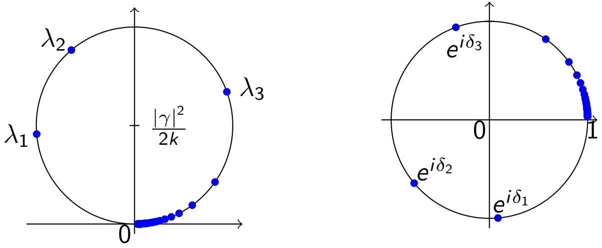


Figure: Left: eigenvalues of $\overline{\mathbf{F}^{\mu}}$. Right: eigenvalues of $\mathbf{S}^{\mu}$.

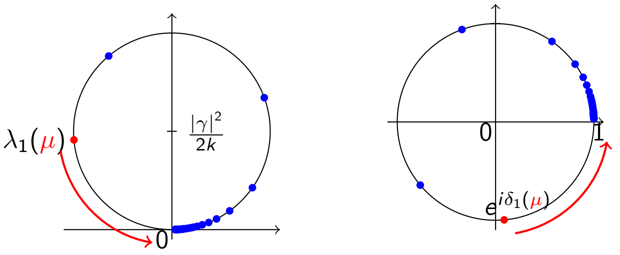
The inside-outside duality method for A.S.E.

$$\mathbf{F}^{\mu} := \overline{\gamma}(S_b^{\mu})^*(F - F_b^{\mu}) = \mathcal{H}_{\mu}^* T_{\mu} \mathcal{H}_{\mu}.$$

Theorem: Assume in addition that $n - 1 \geq 0$ and $\mu > 0$ and is not a A.S.E, then

$$T_{\mu} = T_0 + K_{\mu}$$

with T_0 positive definite and K_{μ} compact.



Theorem: $\mu \to \mu_1(n, D_b)$ with $\mu > \mu_1(n, D_b)$ if and only if $\lambda_1(\mu) \to 0$ or equivalently $\delta_1(\mu) \to 2\pi$.

A numerical illustration

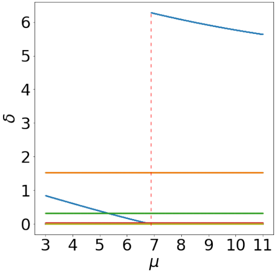


Figure: The curves $\mu \to \delta_m(\mu)$ for D_b a disc or radius $\rho = 0.4333$, $k = 3$ and $n = 1$. The red dashed line indicates the exact A.S.E.

A numerical illustration

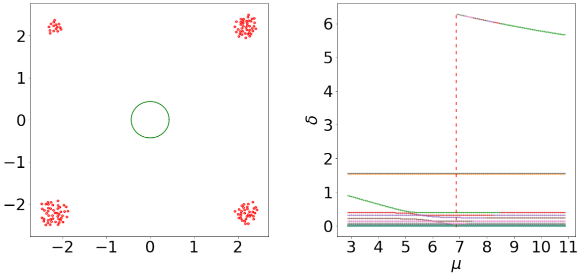


Figure: Left: The domain D (in red) and D_b, the disk of radius $\rho = 0.433$ (in green). Right: Plot of the curves $\mu \to \delta_m(\mu)$ for $k = 3$ and $n = 2$ inside the red discs. The red dashed line indicates the exact A.S.E.

A numerical illustration

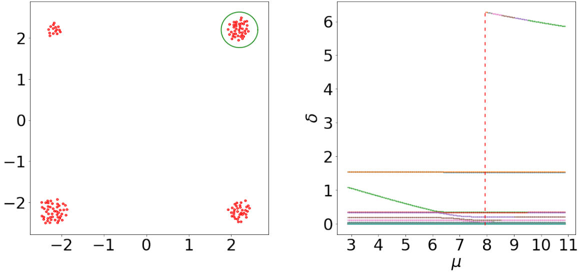
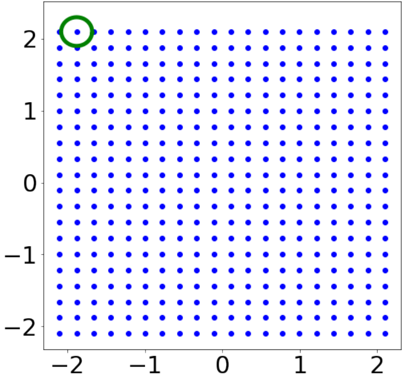


Figure: Left: The domain D (in red) and D_b, the ball of radius $\rho = 0.433$ centered at $(2.2, 2.2)$ (in green). Right: Plot of the curves $\mu \to \delta_m(\mu)$ for $k = 3$ and $n = 2$ inside the red circles. The red dashed line indicates the A.S.E. numerically evaluated by solving the eigenvalue problem.

A first application to the inverse problem

A qualitative type algorithm:

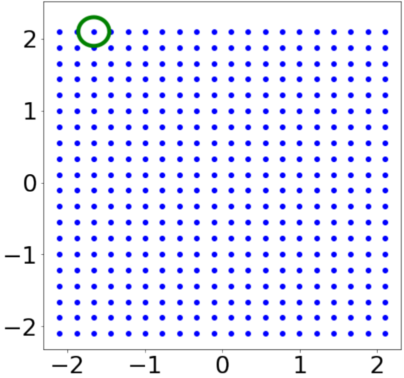
- Sweep a fixed geometry D_b, a ball of radius ρ, over center positions y in a sampling of the probed region.



A first application to the inverse problem

A qualitative type algorithm:

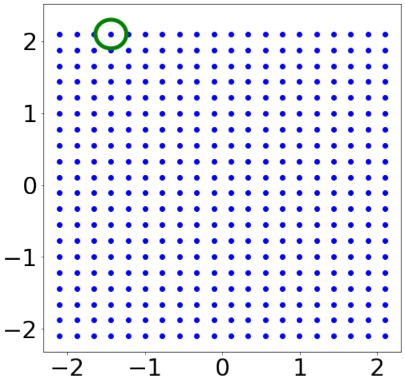
- Sweep a fixed geometry D_b, a ball of radius ρ, over center positions y in a sampling of the probed region.



A first application to the inverse problem

A qualitative type algorithm:

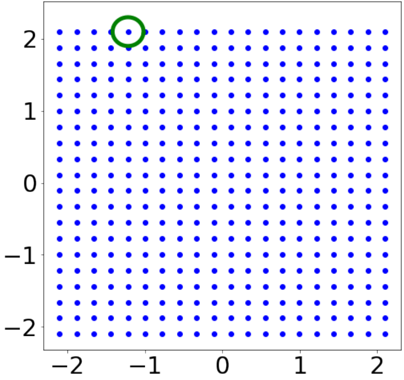
- Sweep a fixed geometry D_b, a ball of radius ρ, over center positions y in a sampling of the probed region.



A first application to the inverse problem

A qualitative type algorithm:

- Sweep a fixed geometry D_b, a ball of radius ρ, over center positions y in a sampling of the probed region.



A first application to the inverse problem

A qualitative type algorithm:

- Sweep a fixed geometry D_b, a ball of radius ρ, over center positions y in a sampling of the probed region.
- For each position of D_b, determine $\mu(n, y)$ from measurements.
- Plot the function $\mathcal{I} : y \to \mu(n, y) - \mu(1, y)$

Remark

- If $n|_{D_b} > 1$, then $\mathcal{I}(y) > 0$
- The larger the $n|_{D_b}$, the larger is $\mathcal{I}(y)$

Numerical illustrations for the first qualitative algorithm

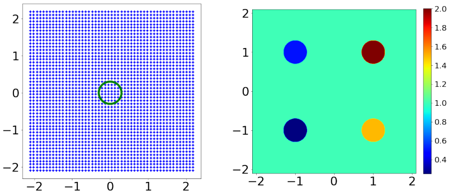


Figure: Four diffracting discs of radius 0.3, $n = 0.25$ (bottom left), $n = 0.5$ (top left), $n = 1.5$ (bottom right) and $n = 2$ (upper right).

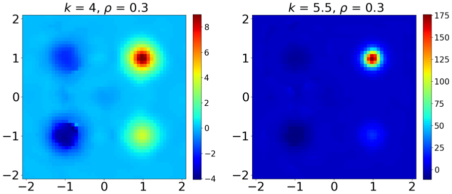


Figure: Indicator function with $\rho = 0.3$ and the noise level 1%.

Numerical illustrations for the first qualitative algorithm

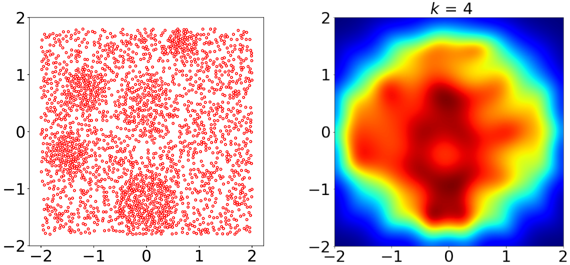


Figure: Reconstruction obtained using the Linear Sampling Method (1% of added noise, $k = 4$).

Numerical illustrations for the first qualitative algorithm

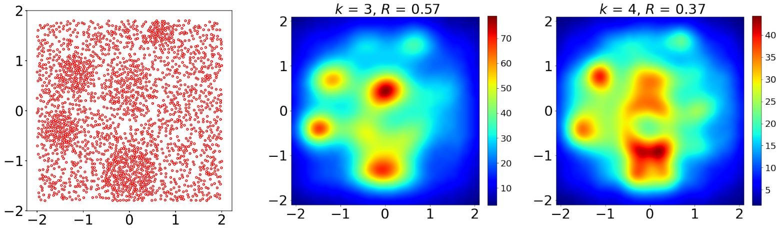


Figure: Reconstructions using the indicator function $y \mapsto I(y)$ (1% of added noise, $k = 3$ (middle) and $k = 4$ (right).

Numerical illustrations for the first qualitative algorithm

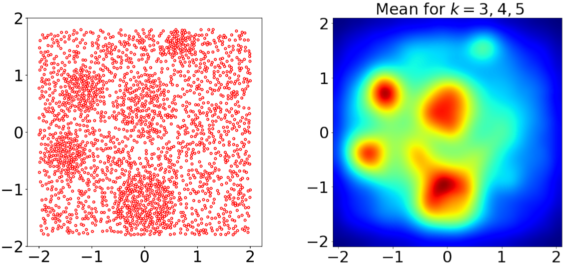


Figure: Mean values of indicator functions for three wavenumbers: $k = 3, 4, 5$

From far field to local macroscopic properties

$$\mu_1(n, D_b) = -\int_{\partial D_b} \partial_\nu w_1 ds = k^2 \int_{D_b} n w_1 dx$$

where $w_1 \in H^1(D_b)$ is the unique solution of

$$\begin{cases} \Delta w_1 + k^2 n w_1 = 0 & \text{in } D_b \\ w_1 = 1 & \text{on } \partial D_b. \end{cases}$$

A classical result from homogenization theory:

Assume that $n = n_\delta \to \overline{n}$ as $\delta \to 0$ weak star in $L^\infty(D_b)$. Then $\mu_1(n_\delta, D_b) \to \mu_1(\overline{n}, D_b)$ as $\delta \to 0$.

From far field to local macroscopic properties

$$\mu_1(n, D_b) = -\int_{\partial D_b} \partial_\nu w_1 ds = k^2 \int_{D_b} n w_1 dx$$

where $w_1 \in H^1(D_b)$ is the unique solution of

$$\begin{cases} \Delta w_1 + k^2 n w_1 = 0 & \text{in } D_b \\ w_1 = 1 & \text{on } \partial D_b. \end{cases}$$

Application: If D_b is a disc of radius ρ and $\overline{n}$ is a constant

$$\mu_1(\overline{n}, D_b) = 2\pi\rho k\sqrt{\overline{n}}\frac{J_1(k\rho\sqrt{\overline{n}})}{J_0(k\rho\sqrt{\overline{n}})}. \qquad (2)$$

$$\overline{n} \mapsto \mu_1(\overline{n}, D_b)$$

is a bijection if $\boxed{\overline{n} \leq \gamma_0/(k\rho)^2}$ where γ_0 is the first zero of J_0.

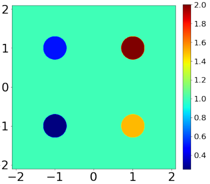
A second application to the inverse problem

A quantitative inversion algorithm:

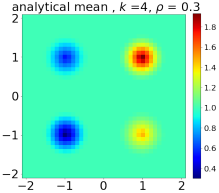
- Choose D_b to be a disc of center y and radius ρ.
- Compute $\mu_1(n, D_b)$ and deduce the constant $\overline{n}(y)$ such that $\mu_1(\overline{n}, D_b) = \mu_1(n, D_b)$

$\overline{n}$ is expected to be an approximation of $\frac{1}{|D_b|} \int_{D_b} n(x)dx$

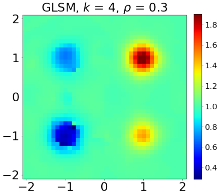
A second application to the inverse problem



Four diffracting discs of radius 0.3



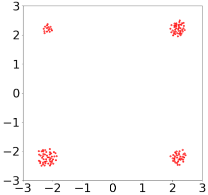
$y \mapsto \frac{1}{|D_b|} \int_{D_b} n(x)dx$



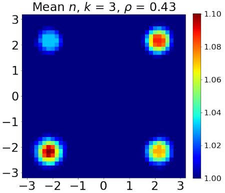
$y \mapsto \overline{n}(y)$

$\mu_1(n, D_b)$ determined from far fields

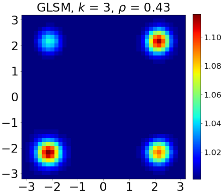
A second application to the inverse problem



$n = 2$ inside the discs of radii 0.02

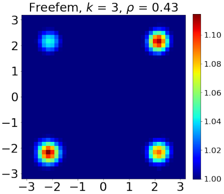


$y \mapsto \frac{1}{|D_b|} \int_{D_b} n(x)dx$



$y \mapsto \overline{n}(y)$

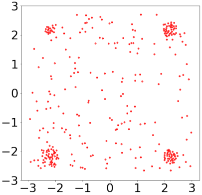
$\mu_1(n, D_b)$ determined from far fields



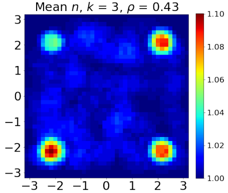
$y \mapsto \overline{n}(y)$

$\mu_1(n, D_b)$ computed numerically

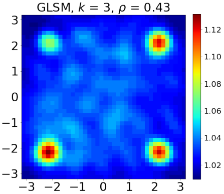
A second application to the inverse problem



$n = 2$ inside the discs of radii 0.02

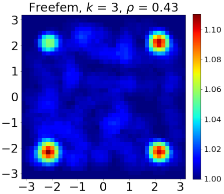


$y \mapsto \frac{1}{|D_b|} \int_{D_b} n(x)dx$



$y \mapsto \overline{n}(y)$

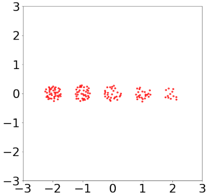
$\mu_1(n, D_b)$ determined from far fields



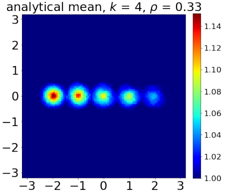
$y \mapsto \overline{n}(y)$

$\mu_1(n, D_b)$ computed numerically

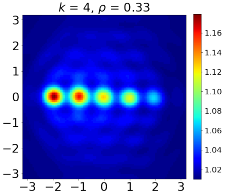
A second application to the inverse problem



Four diffracting discs of radius 0.3



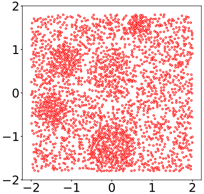
$y \mapsto \frac{1}{|D_b|} \int_{D_b} n(x)dx$



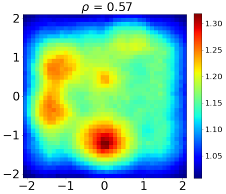
$y \mapsto \overline{n}(y)$

$\mu_1(n, D_b)$ determined from far fields

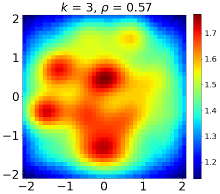
A second application to the inverse problem



$n = 2$ inside the discs of radii 0.02



$y \mapsto \frac{1}{|D_b|} \int_{D_b} n(x)dx$



$y \mapsto \overline{n}(y)$

$\mu_1(n, D_b)$ determined from far fields

A generalization

Let f, be some given function in $H^{1/2}(\partial D_b)$. One can repeat the same arguments replacing the boundary conditions on ∂D_b by

$$\mu u_b + f \int_{\partial D_b} \partial_\nu u_b \cdot \overline{f}\, ds = 0 \text{ on } \partial D_b,$$

The corresponding f-averaged Steklov eigenvalue is $\mu = \mu(n, D_b, f)$ the unique non trivial eigenvalue of

$$\begin{cases} \Delta w + k^2 n w = 0 \text{ in } D_b, \\ \mu w + f \int_{\partial D_b} \partial_\nu w \cdot \overline{f}\, ds = 0 \text{ on } \partial D_b, \end{cases} \tag{3}$$

A generalization

$$\begin{cases} \Delta w + k^2 n w = 0 \text{ in } D_b, \\ \mu w + f \int_{\partial D_b} \partial_\nu w \cdot \overline{f}\, ds = 0 \text{ on } \partial D_b, \end{cases} \tag{3}$$

Theorem: Assume that $k^2 < \eta(n, D_b)$. Then problem (3) has a unique eigenvalue $\mu(n, D_b, f)$.

$$\mu(n, D_b, f) = -\int_{\partial D_b} \partial_\nu w_f \cdot \overline{f}\, ds$$

where $w_f \in H^1(D_b)$ is the unique solution of

$$\begin{cases} \Delta w_f + k^2 n w_f = 0 & \text{in } D_b \\ w_f = f & \text{on } \partial D_b. \end{cases}$$

A generalization

$$\begin{cases} \Delta w + k^2 n w = 0 \text{ in } D_b, \\ \mu w + f \displaystyle\int_{\partial D_b} \partial_\nu w \cdot \overline{f} \, ds = 0 \text{ on } \partial D_b, \end{cases} \tag{3}$$

Theorem: Assume that $k^2 < \eta(n, D_b)$. Then problem (3) has a unique eigenvalue $\mu(n, D_b, f)$.

$$\mu(n, D_b, f) = - \int_{\partial D_b} \partial_\nu w_f \cdot \overline{f} \, ds$$

where $w_f \in H^1(D_b)$ is the unique solution of

$$\begin{cases} \Delta w_f + k^2 n w_f = 0 \quad \text{in } D_b \\ w_f = f \quad \text{on } \partial D_b. \end{cases}$$

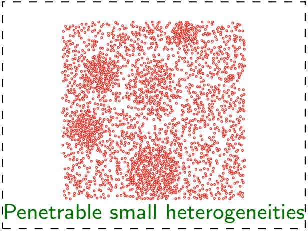
- $\mu(n, D_b, f)$ can be determined from far field data
- Knowing $\mu(n, D_b, f)$ for all f is equivalent to knowing the DtN map $f \mapsto \partial_\nu w_f$

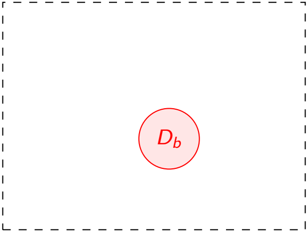
Conclusion / Some perspectives

- We proposed a new imaging algorithm based on the use of averaged Steklov eigenvalues
- We designed two versions of the algorithm: a qualitative one and a quantitative one
- Validation for penetrable scatterers using synthetic data

- Further explore the potentialities of knowing $\mu(n, D_b, f)$
- Extend to qualitative/quantitative evaluation of anisotropy

Modified Transmission Eigenvalues





$$\begin{cases} \Delta u + k^2 n u = 0 \quad \text{in } \mathbb{R}^d \\ n \neq 1 \text{ in } D \text{ and } n = 1 \text{ in } \mathbb{R}^d \setminus D \\ u = u^s + u^i \text{ in } \mathbb{R}^d \setminus \Gamma \\ \lim_{r=|x|\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \end{cases}$$

$$\begin{cases} \Delta u_b + k^2 n_b u_b = 0 \quad \text{in } \mathbb{R}^d \\ n_b = \lambda \text{ in } D_b \text{ and } n_b = 1 \text{ in } \mathbb{R}^d \setminus D_b \\ u_b = u_b^s + u^i \text{ in } \mathbb{R}^d \setminus \overline{D_b} \\ \lim_{r=|x|\to\infty} r\left(\frac{\partial u_b^s}{\partial r} - iku_b^s\right) = 0 \end{cases}$$

The Modified TE are the $\lambda \in \mathbb{C}$ such that there exists a non trivial incident field $u^i = v \in L^2(D_b \cup D)$ such that

$$u_b^s(v) \equiv u^s(v) \text{ in } \mathbb{R}^d \setminus \{D_b \cup D\}.$$

Modified Transmission Eigenvalues

$$u_b^s \equiv u^s \text{ in } \mathbb{R}^d \setminus \{D_b \cup D\}?$$

if and only if $w = u - u_b \in H_0^2(D_b \cup D)$ and $v = u_b \in L^2(D_b \cup D)$

$$\begin{cases} \Delta w + k^2 w = k^2(n_b(\lambda) - n)v & \text{in } D_b \cup D, \\ \Delta v + k^2 n_b(\lambda) v = 0 & \text{in } D_b \cup D \end{cases} \tag{4}$$

In the connected component D_b (assuming that $D_b \cap D = \emptyset$), $w \in H_0^2(D_b)$ and $v \in L^2(D_b)$,

$$\begin{cases} \Delta w + k^2 w = k^2(\lambda - n)v & \text{in } D_b, \\ \Delta v + k^2 \lambda v = 0 & \text{in } D_b \end{cases} \tag{5}$$

Modified Transmission Eigenvalues

$$u_b^s \equiv u^s \text{ in } \mathbb{R}^d \setminus \{D_b \cup D\}?$$

if and only if $w = u - u_b \in H_0^2(D_b \cup D)$ and $v = u_b \in L^2(D_b \cup D)$

$$\begin{cases} \Delta w + k^2 w = k^2(n_b(\lambda) - n)v & \text{in } D_b \cup D, \\ \Delta v + k^2 n_b(\lambda) v = 0 & \text{in } D_b \cup D \end{cases} \tag{4}$$

In the connected component D_b (assuming that $D_b \cap D = \emptyset$), $w \in H_0^2(D_b)$ and $v \in L^2(D_b)$,

$$\begin{cases} \Delta w + k^2 w = k^2(\lambda - n)v & \text{in } D_b, \\ \Delta v + k^2 \lambda v = 0 & \text{in } D_b \end{cases} \tag{5}$$

This is (still) a non self-adjoint eigenvalue problem: $w \in H_0^2(D_b)$

$$(\Delta + k^2\lambda)\frac{1}{\lambda - n}(\Delta + k^2 n)w = 0 \text{ in } D_b$$

A "better" choice for the background

One can use of an artificial metamaterial for the background (as proposed in Audibert-Cakoni-H. (2017))

The background total field verifies

$$\nabla \cdot a_b \nabla u_b + k^2 n_b u_b = 0 \quad \text{in } \mathbb{R}^d$$

$$\begin{array}{|ll|} \hline a_b = n_b = 1 & \text{in } \mathbb{R}^d \setminus D_b \\ a_b = -a < -1 \text{ and } n_b = \lambda \in \mathbb{R} & \text{in } D_b \\ \hline \end{array}$$

A "better" choice for the background

One can use of an artificial metamaterial for the background (as proposed in Audibert-Cakoni-H. (2017))

The background total field verifies

$$\nabla \cdot a_b \nabla u_b + k^2 n_b u_b = 0 \quad \text{in } \mathbb{R}^d$$

$a_b = n_b = 1$	in $\mathbb{R}^d \setminus D_b$
$a_b = -a < -1$ and $n_b = \lambda \in \mathbb{R}$	in D_b

The Modified transmission eigenvalue problem becomes: $u \in H^1(D_b)$, and $v \equiv u_b \in H^1(D_b)$,

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } D_b \\ -a\Delta v + k^2 \lambda v = 0 & \text{in } D_b \\ u = v & \text{on } \partial D_b \\ \dfrac{\partial u}{\partial \nu} = -a \dfrac{\partial v}{\partial \nu} & \text{on } \partial D_b \end{cases}$$

A "better" choice for the background

One can use of an artificial metamaterial for the background (as proposed in Audibert-Cakoni-H. (2017))

The background total field verifies

$$\nabla \cdot a_b \nabla u_b + k^2 n_b u_b = 0 \quad \text{in } \mathbb{R}^d$$

$$\boxed{\begin{array}{ll} a_b = n_b = 1 & \text{in } \mathbb{R}^d \setminus D_b \\ a_b = -a < -1 \text{ and } n_b = \lambda \in \mathbb{R} & \text{in } D_b \end{array}}$$

If n is real valued $\Rightarrow$ Eigenvalue problem for a selfadjoint compact operator: $(u, v) \in \mathcal{H}(D_b)$

$$\int_{D_b} \nabla u \cdot \nabla \overline{u}' \, dx + a \int_{D_b} \nabla v \cdot \nabla \overline{v}' \, dx - k^2 \int_{D_b} n u \overline{u}' \, dx = -k^2 \lambda \int_{D_b} v \overline{v}' \, dx$$

for all $(u', v') \in \mathcal{H}(D_b)$ where

$$\mathcal{H}(D_b) := \left\{ (u, v) \in H^1(D_b) \times H^1(D_b) \text{ such that } u = v \text{ on } \partial D_b \right\}.$$

A "better" choice for the background

Theorem: There is a least one positive eigenvalue $\lambda_a(k,n)$. Assume that $k^2 < \Lambda_0(n, D_b)$ the first eigenvalue of

$$\begin{cases} -\Delta w = \Lambda n w \quad \text{in } D_b \\ w = 0 \quad \text{on } \partial D_b \end{cases}$$

Then the largest positive eigenvalue $\lambda_a(k,n)$ satisfies

$$\lambda_a(k,n) = \sup_{(u,v)\in\mathcal{H}(D_b), v\neq 0} \frac{k^2 \int_{D_b} n|u|^2\,dx - \int_{D_b} (|\nabla u|^2 + a|\nabla v|^2)\,dx}{k^2 \int_{D_b} |v|^2\,dx}.$$

- $\lambda_a(k,n) \to \infty$ as $k^2 \to \Lambda_0(n, D_b)$.
- $n_1 \leq n_2 \Longrightarrow \lambda_a(k,n_1) \leq \lambda_a(k,n_2)$

Limiting problem as $a \to \infty$

Theorem: Assume that $k^2 < \Lambda_0(n, D_b)$. Then,

$$\lambda_a(k,n) \longrightarrow \frac{\mu}{k^2|D_b|} \text{ as } a \to +\infty$$

where $\mu > 0$ is an eigenvalue of the problem $w \in H^1(D_b)$,

$$\begin{cases} \Delta w + k^2 n w = 0 \text{ in } D_b, \\ \mu w + \int_{\partial D_b} \partial_\nu w ds = 0 \text{ on } \partial D_b, \end{cases} \tag{6}$$