Waves and Imaging in Complex Media

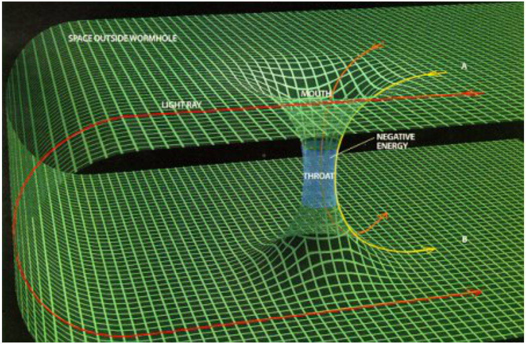
Nonlinearity and Nonlocality Helps to Solve Inverse Problems

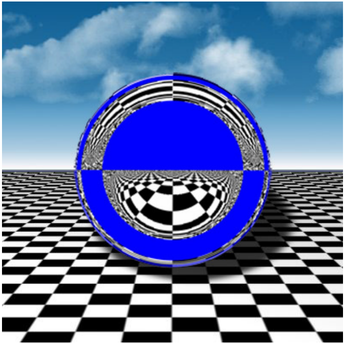
Gunther Uhlmann

University of Washington

Paris, June 10, 2025

Goal: To Determine the Topology and Metric of Space-Time





How can we determine the topology and metric of complicated structures in space-time with a radar-like device?

Figures: Anderson institute and Greenleaf-Kurylev-Lassas-U.

Non-linearity Helps

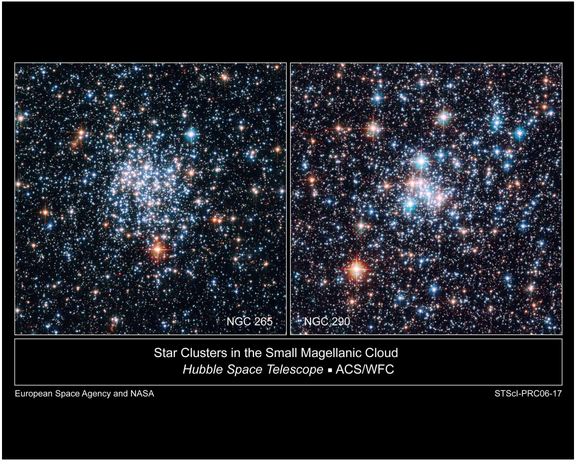
We will consider inverse problems for non-linear wave equations, e.g.
$\frac{\partial^2}{\partial t^2}u(t,y) - c(t,y)^2\Delta u(t,y) + a(t,y)u(t,y)^2 = f(t,y).$

We will show that:

-Non-linearity helps to solve the inverse problem,

-"Scattering" from the interacting wave packets determines the structure of the spacetime.

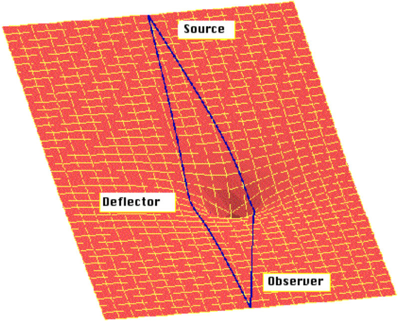
Inverse Problems in Space-Time: Passive Measurements

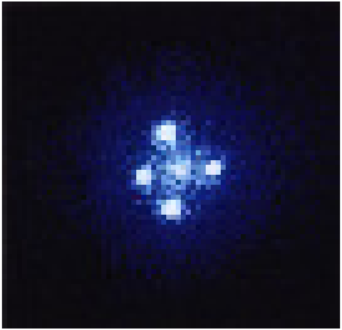


Can we determine the structure of space-time when we see light coming from many point sources varying in time? We can also observe gravitational waves.

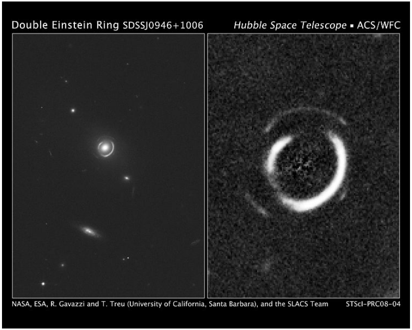
Gravitational Lensing

We consider e.g. light or X-ray observations or measurements of gravitational waves.

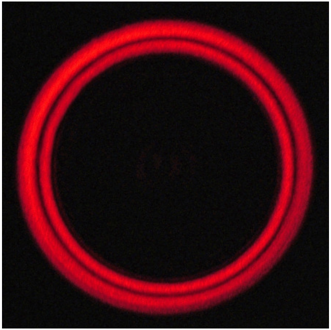




Gravitational Lensing



Double Einstein Ring

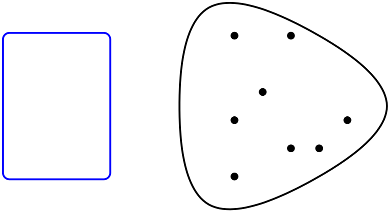


Conical Refraction

Passive Measurements: Gravitational Waves

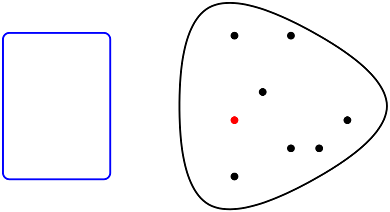
NSF Announcement, Feb 11, 2015

Inverse Problem for Passive Measurements



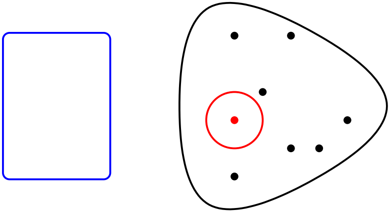
Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Inverse Problem for Passive Measurements



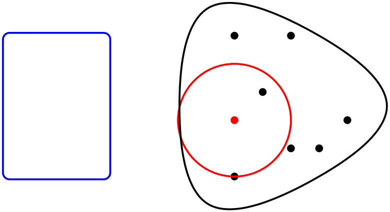
Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Inverse Problem for Passive Measurements



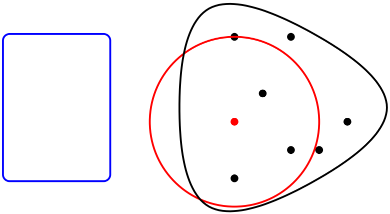
Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Inverse Problem for Passive Measurements



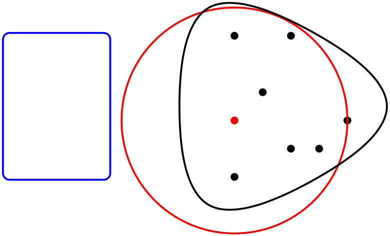
Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Inverse Problem for Passive Measurements



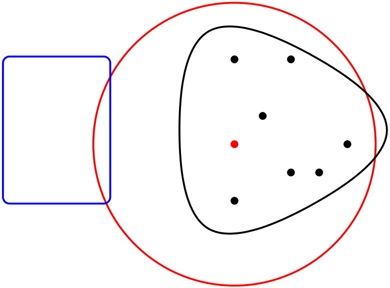
Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Inverse Problem for Passive Measurements



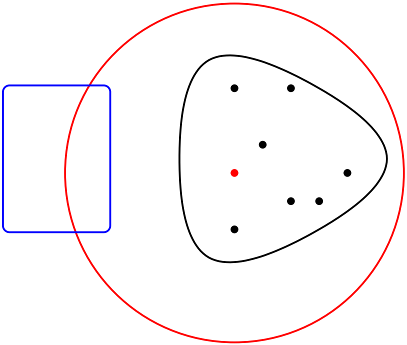
Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Inverse Problem for Passive Measurements



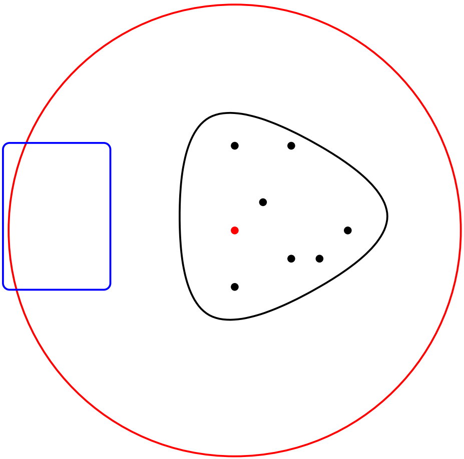
Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Inverse Problem for Passive Measurements



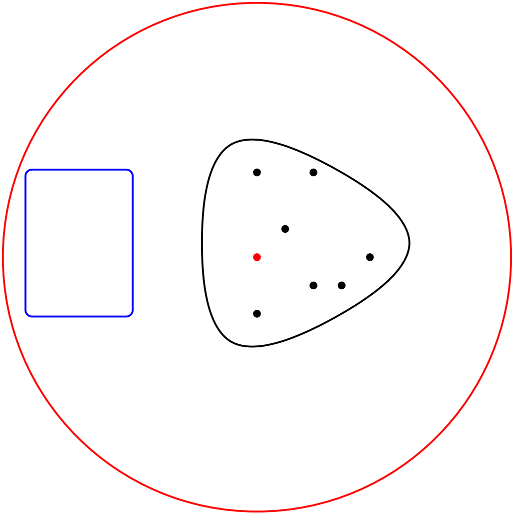
Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Inverse Problem for Passive Measurements



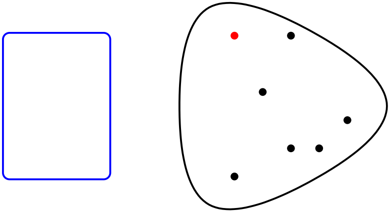
Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Inverse Problem for Passive Measurements



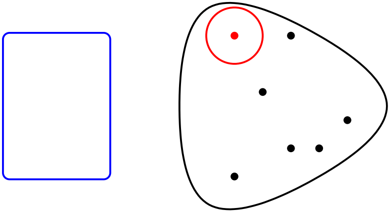
Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Inverse Problem for Passive Measurements



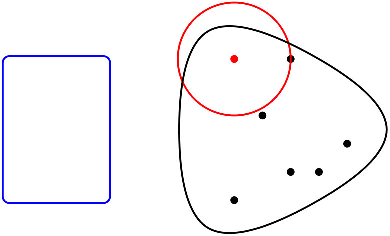
Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Inverse Problem for Passive Measurements



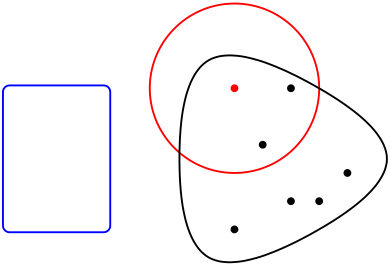
Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Inverse Problem for Passive Measurements



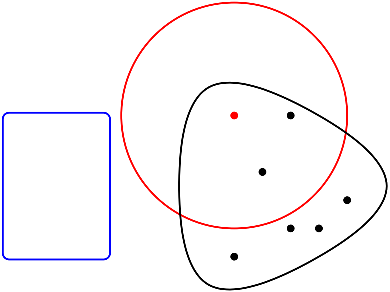
Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Inverse Problem for Passive Measurements



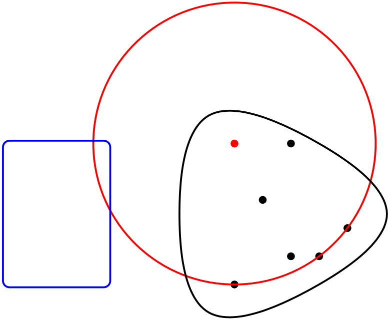
Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Inverse Problem for Passive Measurements



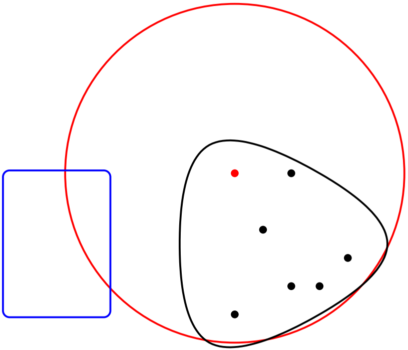
Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Inverse Problem for Passive Measurements



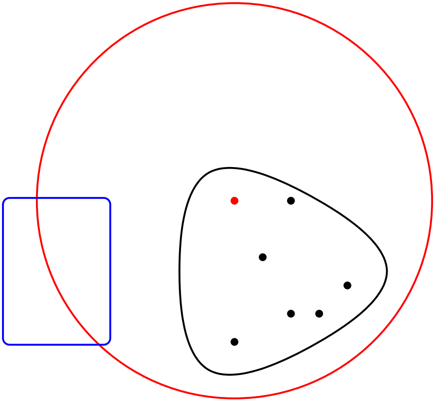
Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Inverse Problem for Passive Measurements



Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Inverse Problem for Passive Measurements



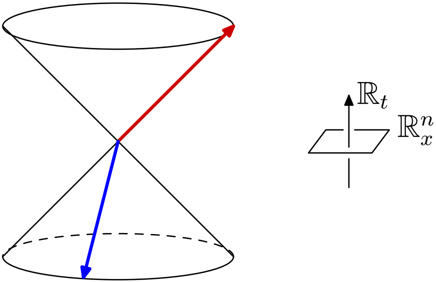
Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Lorentzian Geometry

$(n+1)$-dimensional Minkowski space: (M, g)

$$M = \mathbb{R}^{1+n} = \mathbb{R}_t \times \mathbb{R}^n_x, \quad \text{metric: } g = -dt^2 + dx^2.$$

Null/lightlike vectors: $V \in T_q M$ with $g(V, V) = 0$.



$L^{\pm}_q M$: future/past null vectors

Lorentzian Geometry

In general:

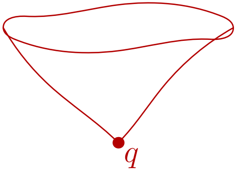
$M = (n+1)$-dimensional manifold, g Lorentzian $(-,+,\ldots,+)$.

Assume: existence of time orientation.

$$T_q M \cong (\mathbb{R}^{1+n}, \text{Minkowski metric}).$$

Null-geodesics: $\gamma(s) = \exp_q(sV)$, $V \in T_q M$ null.

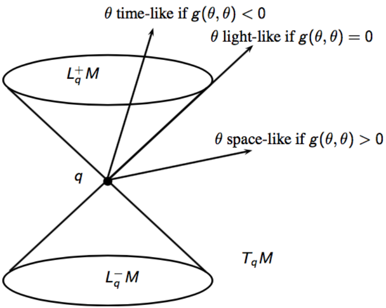
Future light cone: $\mathcal{L}_q^+ = \{\exp_q(V) \colon V \text{ future null}\}$



Lorentzian Manifolds

Let (M, g) be a $1 + 3$ dimensional time oriented Lorentzian manifold.
The signature of g is $(-, +, +, +)$.
Example: Minkowski space-time $(\mathbb{R}^4, g_m)$, $g_m = -dt^2 + dx^2 + dy^2 + dz^2$.

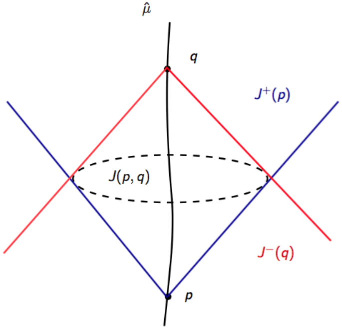
- $L_q^{\pm} M$ is the set of future (past) pointing light like vectors at q.
- Casual vectors are the collection of time-like and light-like vectors.
- A curve γ is time-like (light-like, causal) if the tangent vectors are time-like (light-like, causal).



Causal Relations

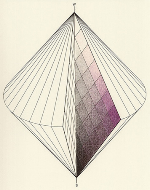
Let $\widehat{\mu}$ be a time-like geodesic, which corresponds to the world-line of an observer in general relativity. For $p, q \in M$, $p \ll q$ means p, q can be joined by future pointing time-like curves, and $p < q$ means p, q can be joined by future pointing causal curves.

- The chronological future of $p \in M$ is $I^+(p) = \{q \in M : p \ll q\}$.
- The causal future of $p \in M$ is $J^+(p) = \{q \in M : q < p\}$.
- $J(p, q) = J^+(p) \cap J^-(q)$, $I(p, q) = I^+(p) \cap I^-(q)$.



Global Hyperbolicity

A Lorentzian manifold (M, g) is globally hyperbolic if



- there is no closed causal paths in M;
- for any $p, q \in M$
  and $p < q$, the set $J(p, q)$ is compact.

Then hyperbolic equations are well-posed on (M, g)
Also, (M, g) is isometric to the product manifold

$$\mathbb{R} \times N \text{ with } g = -\beta(t, y)dt^2 + \kappa(t, y).$$

Here $\beta : \mathbb{R} \times N \to \mathbb{R}_+$ is smooth, N is a 3 dimensional manifold and κ is a Riemannian metric on N and smooth in t.
We shall use $x = (t, y) = (x_0, x_1, x_2, x_3)$ as the local coordinates on M.

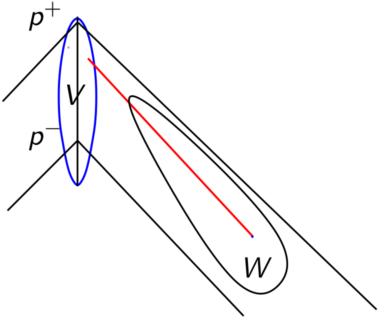
Light Observation Set

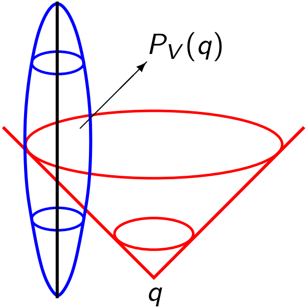
Let $\mu = \mu([-1, 1]) \subset M$ be time-like geodesics containing p^- and p^+. We consider observations in a neighborhood $V \subset M$ of μ.

Let $W \subset I^-(p^+) \setminus J^-(p^-)$ be relatively compact and open set.

The light observation set for $q \in W$ is

$$P_V(q) := \{\gamma_{q,\xi}(r) \in V;\ r \geq 0,\ \xi \in L^+_q M\}.$$





Inverse Problems with Passive Measurements

The earliest light observation set of $q \in M$ in V is

$$\mathcal{E}_V(q) = \{x \in \mathcal{P}_V(q) : \text{ there is no } y \in \mathcal{P}_V(q) \text{ and future pointing} \\ \text{time like path } \alpha \text{ such that } \alpha(0) = y \text{ and } \alpha(1) = x\} \subset V.$$

In the physics literature the light observation sets are called light-cone cuts (Engelhardt-Horowitz, arXiv 2016)

Theorem (Kurylev-Lassas-U 2018, arXiv 2014)

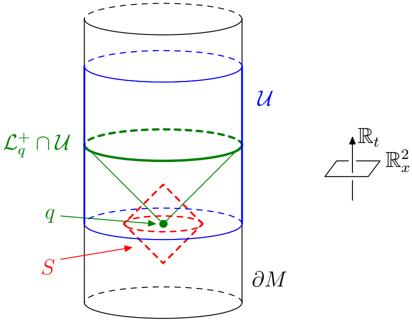
Let (M, g) be an open smooth globally hyperbolic Lorentzian manifold of dimension $n \geq 3$ and let $p^+, p^- \in M$ be the points of a time-like geodesic $\widehat{\mu}([-1,1]) \subset M$, $p^\pm = \widehat{\mu}(s_\pm)$. Let $V \subset M$ be a neighborhood of $\widehat{\mu}([-1,1])$ and $W \subset M$ be a relatively compact set. Assume that we know

$$\mathcal{E}_V(W).$$

Then we can determine the topological structure, the differential structure, and the conformal structure of W, up to diffeomorphism.

Boundary Light Observation Set

$M = \{(t,x)\colon |x| < 1\} \subset \mathbb{R}^{1+2}$.



Set of sources $S \subset M^\circ$.

Observations in $\mathcal{U} \subset \partial M$.

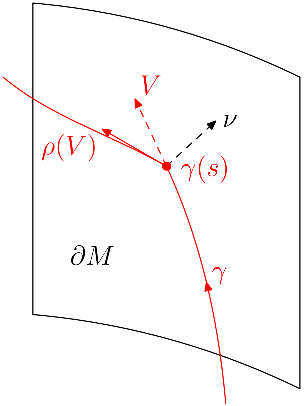
Data: $\mathscr{S} = \{\mathcal{L}_q^+ \cap \mathcal{U} \colon q \in S\}$

Theorem

The collection $\mathscr{S}$ determines the topological, differentiable, and conformal structure $[g|_S] = \{fg|_S \colon f > 0\}$ of S.

Reflection at the Boundary

γ null-geodesic until $\gamma(s) \in \partial M$.



$\rho(V) =$ reflection of V across ∂M. (Snell's law.)

$\to$ continuation of γ as broken null-geodesic

Null-convexity

Simplest case:

$$\text{All null-geodesics starting in } M^\circ \text{ hit } \partial M \text{ transversally.} \qquad (1)$$

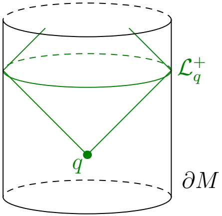
Proposition

(1) *is equivalent to null-convexity of* ∂M*:*

$$II(W,W) = g(\nabla_W \nu, W) \geq 0, \quad W \in T\partial M \text{ null.}$$

Stronger notion: strict null-convexity. ($II(W,W) > 0$, $W \neq 0$.)

Define light cones $\mathcal{L}_q^+$ using broken null-geodesics.



Main Result

Setup:

- (M, g) Lorentzian, dim ≥ 2, strictly null-convex boundary
- existence of $t \colon M \to \mathbb{R}$ proper, timelike
- sources: $S \subset M^\circ$ with $\bar{S}$ compact
- observations in $\mathcal{U} \subset \partial M$ open

Assumptions:

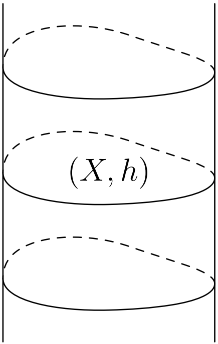
1. $\mathcal{L}^+_{q_1} \cap \mathcal{U} \neq \mathcal{L}^+_{q_2} \cap \mathcal{U}$ for $q_1 \neq q_2 \in \bar{S}$
2. points in S and $\mathcal{U}$ are not (null-)conjugate

Theorem (Hintz–U, 2019)

The smooth manifold $\mathcal{U}$ and the unlabelled collection $\mathscr{S} = \{\mathcal{L}^+_q \cap \mathcal{U} \colon q \in S\} \subset 2^{\mathcal{U}}$ uniquely determine $(S, [g|_S])$ (topologically, differentiably, and conformally).

Example for (M, g)

(X, h) compact Riemannian manifold with boundary.

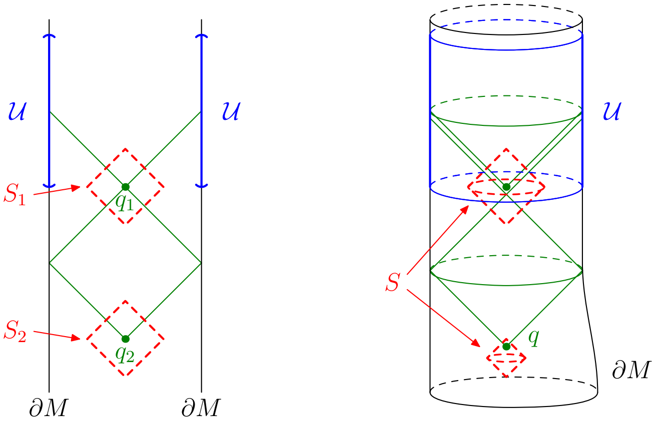


$$M = \mathbb{R}_t \times X, \quad g = -dt^2 + h.$$

(Strict) null-convexity of $\partial M \iff$ (strict) convexity of ∂X

'Counterexamples'

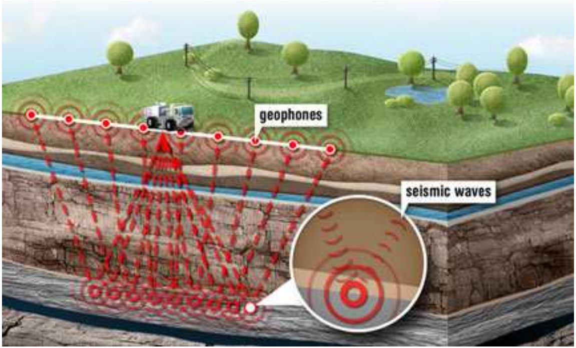
Necessity of assumption 1. ($\mathcal{L}^+_{q_1} \cap \mathcal{U} \neq \mathcal{L}^+_{q_2} \cap \mathcal{U}$ for $q_1 \neq q_2 \in \bar{S}$)



S_1 and $S_1 \cup S_2$ are indistinguishable from $\mathcal{U}$.

Inverse Problems for Linear Hyperbolic Equations

- Rakesh-Symes 1987: Inverse problem for $\partial_t^2 - \Delta + q$.
- Belishev-Kurylev 1992 and Tataru 1995: Reconstruction of a Riemannian manifold with time-independent metric.
- Unique continuation needed for Belishev-Kurylev-Tataru results fail for time-depending wave speed.



Active Measurements

Wave equation: Let $g=[g_{jk}(y)]_{j,k=1}^n$ and $u=u^f(y,t)$ be the solution of

$$
\begin{aligned}
&(\partial_t^2 u-\Delta_g)u=0 \quad \text{on } N\times\mathbb{R}_+,\\
&u|_{\partial N\times\mathbb{R}_+}=f,\\
&u|_{t=0}=0,\quad u_t|_{t=0}=0.
\end{aligned}
$$

Here N is a compact Riemannian manifold with boundary, ν is the unit normal of ∂N,

$$\Delta_g u=\sum_{j,k=1}^n |g|^{-1/2}\frac{\partial}{\partial y^j}(|g|^{1/2}g^{jk}\frac{\partial}{\partial y^k}u),$$

where $|g|=\det(g_{ij})$ and $[g_{ij}]=[g^{jk}]^{-1}$. Let

$$\Lambda f=\partial_\nu u^f|_{\partial N\times\mathbb{R}_+}.$$

We are given boundary data $(\partial N,\Lambda)$.

Active Measurements

$$
\begin{aligned}
&(\partial_t^2 u - \Delta_g) u = 0 \quad \text{on } N \times \mathbb{R}_+, \\
&u|_{\partial N \times \mathbb{R}_+} = f, \\
&u|_{t=0} = 0, \quad u_t|_{t=0} = 0.
\end{aligned}
\qquad \Lambda f = \partial_\nu u^f|_{\partial N \times \mathbb{R}_+}
$$

Inverse Problem: Can we recover g from Λ up to an isometry, which is an identity at boundary?

Theorem (Belishev-Kurylev 1992, Tataru 1995): This is true.

- need g to be independent of t;
- Tataru's result is only valid for metrics that depend analytically on t.
- The Belishev-Kurylev-Tataru result has been extended by Eskin (2017) to metrics that are real-analytic in the time variable.

Geometrical Optics

Let $q \in C_0^\infty(\mathbb{R}^n)$, $\mathit{supp}\, q \subset \{x \in \mathbb{R}^n : |x| < R\}$.
Let $\omega \in S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

$$CP \quad \begin{cases} ((\partial_t^2 - \Delta) + q)u = & 0 \ \mathit{on}\, \mathbb{R}_x^n \times \mathbb{R}_t \\ \\ u = & \delta(t - x \cdot \omega),\ t < -R \end{cases}$$

$$\langle \delta(t - x \cdot \omega), \varphi \rangle = \int_{x \cdot \omega = t} \varphi(x)\, dH, \qquad \varphi \in C_0^\infty(\mathbb{R}^n)$$

Progressing Waves

$\delta(t - x \cdot \omega)$ solves

$$\Box \delta(t - x \cdot \omega) = 0$$

where $\Box = \partial_t^2 - \Delta$ is the D'Alembertian.

$$(\Box + q)\delta(t - x \cdot \omega) = q\delta(t - x \cdot \omega)$$

Next try

$$u_1(t, x, \omega) = \delta(t - x \cdot \omega) + a_1(x, \omega)H(t - x \cdot \omega)$$

$$H(t - x \cdot \omega) = \begin{cases} 1 & t > x \cdot \omega \\ 0 & t < x \cdot \omega \end{cases}$$

$$\Box H(t - x \cdot \omega) = 0$$

Progressing Waves

$$(\Box + q)u_1 = (q(x) + 2\nabla a_1 \cdot \omega)\delta(t - x\cdot\omega)$$

$$+ (q(x)a_1 - \Delta a_1)H(t - x\cdot\omega)$$

To eliminate main singularity, we choose

$$\nabla a_1 \cdot \omega = -\frac{q(x)}{2}$$

$$a_1(x,\omega) = -\frac{1}{2}\int_{-\infty}^{x\cdot\omega} q(x + (s - x\cdot\omega)\omega)ds$$

Progressing Waves

If $x \cdot \omega > R$,

$$a_1(x, \omega) = \text{ X-ray transform of } -q/2$$

Next try

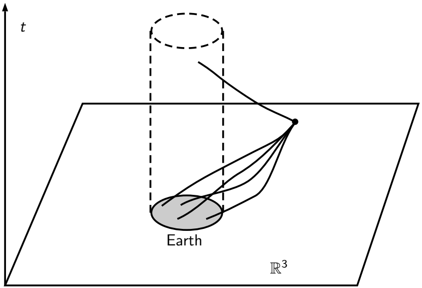
$$If(x, \omega) = \int f(x + s\omega)ds, \quad f \in C_0^\infty(\mathbb{R}^n)$$

$$u_2 = \delta(t - x \cdot \omega) + a_1(x, \omega)H(t - x \cdot \omega) + a_2(x, \omega)(t - x \cdot \omega)_+$$

$$\textit{where} \quad s_+^k = \begin{cases} s^k & s > 0 \\ 0 & s < 0 \end{cases} \quad \textit{and} \quad a_2 \in C^\infty(\mathbb{R}^n \times S^{n-1})$$

$$\nabla a_2 \cdot \omega = -\frac{1}{2}(q(x)a_1 - \Delta a_1)$$

Interaction of Nonlinear Waves



Inverse Problem for a Non-linear Wave Equation

Consider the non-linear wave equation

$$\Box_g u(x) + a(x)\, u(x)^2 = f(x) \quad \text{on } M^0 = (-\infty, T) \times N,$$
$$\text{supp }(u) \subset J_g^+(\text{supp }(f)),$$

where $\text{supp}(f) \subset V$, $V \subset M$ is open,

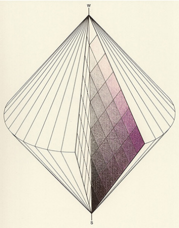
$$\Box_g u = -\sum_{p,q=1}^{4} (-\det(g(x)))^{-1/2} \frac{\partial}{\partial x^p} \left((-\det(g(x)))^{1/2} g^{pq}(x) \frac{\partial}{\partial x^q} u(x) \right),$$

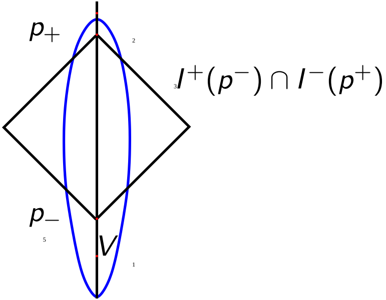
$\det(g) = \det((g_{pq}(x))_{p,q=1}^4)$, $f \in C_0^6(V)$ is a controllable source, and $a(x)$ is a non-vanishing C^∞-smooth function.
In a neighborhood $\mathcal{W} \subset C_0^2(V)$ of the zero-function, define the measurement operator by

$$L_V : f \mapsto u|_V, \quad f \in C_0^6(V).$$

Theorem (Kurylev-Lassas-U, 2018)

Let (M, g) be a globally hyperbolic Lorentzian manifold of dimension $(1 + 3)$. Let μ be a time-like path containing p^- and p^+, $V \subset M$ be a neighborhood of μ, and $a : M \to \mathbb{R}$ be a non-vanishing function. Then $(V, g|_V)$ and the measurement operator L_V determines the set $I^+(p^-) \cap I^-(p^+) \subset M$ and the conformal class of the metric g, up to a change of coordinates, in $I^+(p^-) \cap I^-(p^+)$.

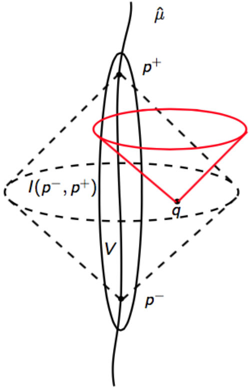




Idea of the Proof in the Case of Quadratic Nonlinearity: Interaction of Singularities

We construct the earliest light observation set by producing artificial point sources in $I(p_-, p_+)$. The key is the singularities generated from nonlinear interaction of linear waves.

- We construct sources f so that the solution u has new singularities.
- We characterize the type of the singularities.
- We determine the order of the singularities and find the principal symbols.



Non-linear Geometrical Optics

Let $u = \varepsilon w_1 + \varepsilon^2 w_2 + \varepsilon^3 w_3 + \varepsilon^4 w_4 + E_\varepsilon$ satisfy

$$\Box_g u + a u^2 = f, \quad \text{in } M^0 = (-\infty, T) \times N,$$
$$u|_{(-\infty,0)\times N} = 0$$

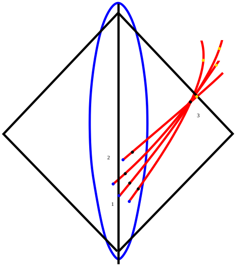
with $f = \varepsilon f_1$. When $Q = \Box_g^{-1}$, we have

$$\begin{aligned}
w_1 &= Qf, \\
w_2 &= -Q(a\, w_1\, w_1), \\
w_3 &= 2Q(a\, w_1\, Q(a\, w_1\, w_1)), \\
w_4 &= -Q(a\, Q(a\, w_1\, w_1)\, Q(a\, w_1\, w_1)) \\
&\quad -4Q(a\, w_1\, Q(a\, w_1\, Q(a\, w_1\, w_1))), \\
\|E_\varepsilon\| &\le C\varepsilon^5.
\end{aligned}$$

Non-linear Geometrical Optics

The product has, in a suitable microlocal sense, a principal symbol.

There is a lot of technology availale for the interaction analysis of conormal waves: intersecting pairs of conormal distributions (Melrose-U, 1979, Guillemin-U, 1981, Greenleaf-U, 1991).



Interaction of Waves in Minkowski Space $\mathbb{R}^4$

Let x^j, $j = 1, 2, 3, 4$ be coordinates such that $\{x^j = 0\}$ are light-like. We consider waves

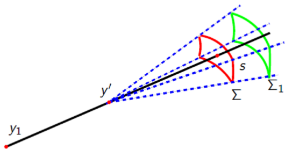
$$\begin{aligned} u_j(x) &= v \cdot (x^j)_+^m, \quad (s)_+^m = |s|^m H(s), \quad v \in \mathbb{R}, j = 1, 2, 3, 4. \\ x^j &= t - x \cdot \omega_j, \quad |\omega_j| = 1 \end{aligned}$$

Waves u_j are conormal distributions, $u_j \in I^{m+1}(K_j)$, where

$$K_j = \{x^j = 0\}, \quad j = 1, 2, 3, 4.$$

The interaction of the waves $u_j(x)$ produce new sources on

$$\begin{aligned} K_{12} &= K_1 \cap K_2, \\ K_{123} &= K_1 \cap K_2 \cap K_3 = \text{line}, \\ K_{1234} &= K_1 \cap K_2 \cap K_3 \cap K_4 = \{q\} = \text{one point}. \end{aligned}$$



Interaction of Two Waves (Second order linearization)

If we consider sources $f_{\vec{\varepsilon}}(x) = \varepsilon_1 f_{(1)}(x) + \varepsilon_2 f_{(2)}(x)$, $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2)$, and the corresponding solution $u_{\vec{\varepsilon}}$, we have

$$\begin{aligned} W_2(x) &= \frac{\partial}{\partial \varepsilon_1} \frac{\partial}{\partial \varepsilon_2} u_{\vec{\varepsilon}}(x)|_{\vec{\varepsilon}=0} \\ &= Q(a\, u_{(1)} \cdot u_{(2)}), \end{aligned}$$

where $Q = \Box_g^{-1}$ and

$$u_{(j)} = Q f_{(j)}.$$

Recall that $K_{12} = K_1 \cap K_2 = \{x^1 = x^2 = 0\}$. Since the normal bundle N^*K_{12} contain only light-like directions $N^*K_1 \cup N^*K_2$,

$$\mathrm{singsupp}(W_2) \subset K_1 \cup K_2.$$

Thus no new interesting singularities are produced by the interaction of two waves (Greenleaf-U, 1991).

Three plane waves interact and produce a conic wave. (Bony, 1996, Melrose-Ritter, 1987, Rauch-Reed, 1982)

Interaction of Three Waves (Third order linearization)

If we consider sources $f_{\vec{\varepsilon}}(x) = \sum_{j=1}^{3} \varepsilon_j f_{(j)}(x)$, $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, and the corresponding solution $u_{\vec{\varepsilon}}$, we have

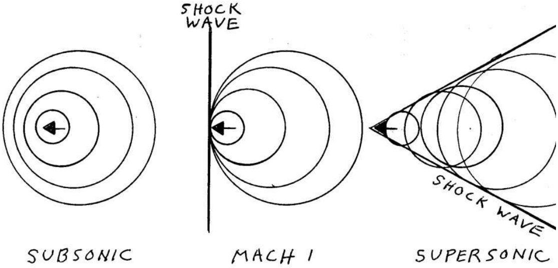
$$\begin{aligned} W_3 &= \partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3} u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} \\ &= \quad 4Q(a\, u_{(1)}\ Q(a\, u_{(2)}\ u_{(3)})) \\ &\quad +4Q(a\, u_{(2)}\ Q(a\, u_{(1)}\ u_{(3)})) \\ &\quad +4Q(a\, u_{(3)}\ Q(a\, u_{(1)}\ u_{(2)})), \end{aligned}$$

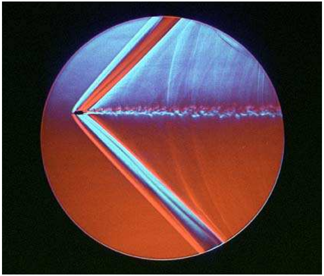
where $Q = \square_g^{-1}$. The interaction of the three waves happens on the line $K_{123} = K_1 \cap K_2 \cap K_3$.

The normal bundle N^*K_{123} contains light-like directions that are not in $N^*K_1 \cup N^*K_2 \cup N^*K_3$ and hence new singularities are produced.

Interaction of Waves

The non-linearity helps in solving the inverse problem.
Artificial sources can be created by interaction of waves using the non-linearity of the wave equation.



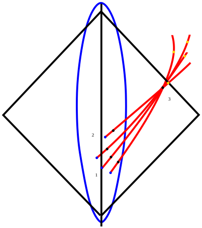


The interaction of 3 waves creates a point source in space that seems to move at a higher speed than light, that is, it appears like a tachyonic point source, and produces a new "shock wave" type singularity.

Interaction of Four Waves

The 3-interaction produces conic waves (only one is shown below).

The 4-interaction produces a spherical wave from the point q that determines the light observation set $P_V(q)$.



Interaction of Four Waves (Fourth order linearization)

If we consider sources $f_{\vec{\varepsilon}}(x) = \sum_{j=1}^{4} \varepsilon_j f_{(j)}(x)$, $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$, and the corresponding solution $u_{\vec{\varepsilon}}$, we have following. Consider

$$W_4 \quad = \quad \partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3}\partial_{\varepsilon_4} u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}.$$

Since $K_{1234} = \{q\}$ we have $N^* K_{1234} = T_q^* M$. Hence new singularities are produced and

$$\mathrm{singsupp}(W_4) \subset (\cup_{j=1}^{4} K_j) \cup \Sigma \cup \mathcal{L}_q^+ M,$$

where Σ is the union of conic waves produced by sources on K_{123}, K_{134}, K_{124}, and K_{234}. Moreover, $\mathcal{L}_q^+ M$ is the union of future going light-like geodesics starting from the point q.

Active and Passive Measurements

(M, g) $(2+1)$-dimensional, $\Box_g u = u^3 + f$.

Idea (Kurylev-Lassas-U 2018, arXiv 2014): Using nonlinearity to create point sources in $I(p_-, p_+)$.

$$f = \sum_{i=1}^{3} \epsilon_i f_i, \quad u_i := \Box_g^{-1} f_i.$$

Take f_i = conormal distribution, e.g.

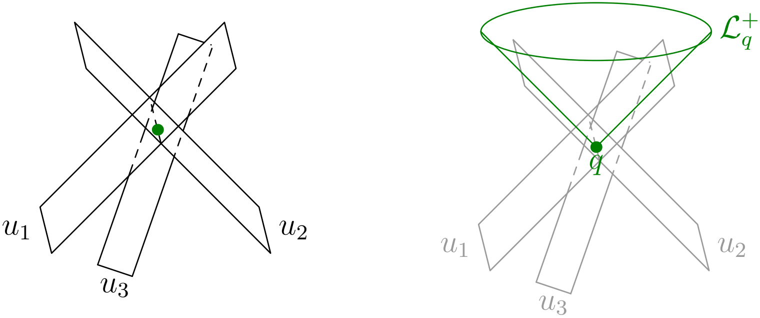
$$f_1(t,x) = (t - x_1)_+^{11} \chi(t,x), \quad \chi \in C_c^\infty(\mathbb{R}^{1+2}).$$

Then

$$u \approx \sum \epsilon_i u_i + 6\epsilon_1\epsilon_2\epsilon_3 \Box_g^{-1}(u_1 u_2 u_3).$$

Generating Point Sources

non-linear interaction of conormal waves $u_i = \Box_g^{-1} f_i$: $\Box_g^{-1}(u_1 u_2 u_3)$



$$q = \bigcap_{i=1}^{3} \mathit{sing\ supp}\ u_i, \quad \mathcal{L}_q^+ = \mathit{sing\ supp}\ \Box_g^{-1}(u_1 u_2 u_3)$$

$\Rightarrow$ singularities of $\partial^3_{\epsilon_1\epsilon_2\epsilon_3} u$ give light observation sets $\mathcal{L}_q^+$

Active and Passive Measurements

(M,g) $(2+1)$-dimensional, $\Box_g u = a(x)u^3 + f$, $a \neq 0$.

Idea (Kurylev-Lassas-U 2018, arXiv 2014): Using nonlinearity to create point sources in $I(p_-, p_+)$.

$$f = \sum_{i=1}^{3} \epsilon_i f_i, \quad u_i := \Box_g^{-1} f_i.$$

Take $f_i =$ conormal distribution, e.g.

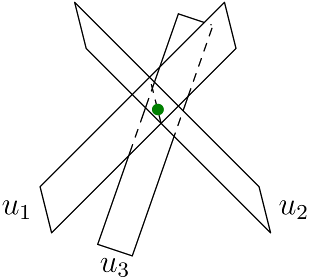
$$f_1(t,x) = (t - x_1)_+^{11} \chi(t,x), \quad \chi \in C_c^\infty(\mathbb{R}^{1+2}).$$

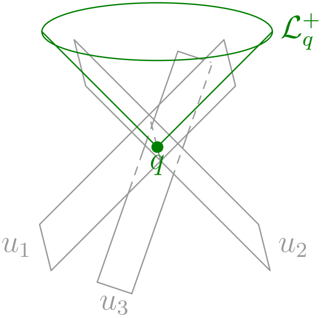
Then

$$u \approx \sum \epsilon_i u_i + 6\epsilon_1\epsilon_2\epsilon_3 \Box_g^{-1}(u_1 u_2 u_3).$$

Generating Point Sources

non-linear interaction of conormal waves $u_i = \Box_g^{-1} f_i$: $\Box_g^{-1}(u_1 u_2 u_3)$





$$q = \bigcap_{i=1}^{3} \mathit{sing\ supp}\ u_i, \quad \mathcal{L}_q^+ = \mathit{sing\ supp}\ \Box_g^{-1}(u_1 u_2 u_3)$$

$\Rightarrow$ singularities of $\partial^3_{\epsilon_1 \epsilon_2 \epsilon_3} u$ give light observation sets $\mathcal{L}_q^+$

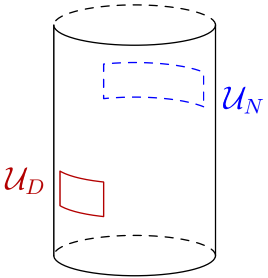
Active Measurements for Boundary Value Problems

Theorem (Hintz-U-Zhai, 2021)

Model (in $\dim M = 1 + 2$)

$$\Box_g u = a(x)u^3,\ a \neq 0, \quad u|_{\mathcal{U}_D} = u_0 \in \mathcal{C}_c^{10}(\mathcal{U}_D).$$

Measure $L\colon u_0 \mapsto \partial_\nu u|_{\mathcal{U}_N}$. Recover a and g from L.



Propagation of singularities: (strict) null-convexity assumption simplifies structure of null-geodesic flow. (Taylor '75, '76, Melrose–Sjöstrand '78, '82.)

(Special case: $\mathcal{U}_N = \mathcal{U}_D$.)

Einstein's Equations

The Einstein equation for the $(-,+,+,+)$-type Lorentzian metric g_{jk} of the space time is

$$\mathrm{Ein}_{jk}(g) = T_{jk},$$

where

$$\mathrm{Ein}_{jk}(g) = \mathrm{Ric}_{jk}(g) - \frac{1}{2}(g^{pq}\,\mathrm{Ric}_{pq}(g))g_{jk}.$$

In vacuum, $T=0$. In wave map coordinates, the Einstein equation yields a quasilinear hyperbolic equation and a conservation law,

$$g^{pq}(x)\frac{\partial^2}{\partial x^p \partial x^q} g_{jk}(x) + B_{jk}(g(x), \partial g(x)) = T_{jk}(x),$$
$$\nabla_p(g^{pj}\,T_{jk}) = 0.$$

Einstein's Equations Coupled with Matter Fields

$$\begin{aligned}
&\mathrm{Ein}(g) = T, \quad T = \mathrm{T}(\phi, g) + \mathcal{F}_1, \quad \text{on } (-\infty, T) \times N,\\
&\square_g \phi_\ell - m^2 \phi_\ell = \mathcal{F}_2^\ell, \quad \ell = 1, 2, \dots, L,\\
&g|_{t<0} = \widehat{g}, \quad \phi|_{t<0} = \widehat{\phi}.
\end{aligned}$$

Here, $\widehat{g}$ and $\widehat{\phi}$ are C^∞-smooth and satisfy the equations above with zero sources and

$$\mathrm{T}_{jk}(g, \phi) = \sum_{\ell=1}^{L} \partial_j \phi_\ell \, \partial_k \phi_\ell - \frac{1}{2} g_{jk} g^{pq} \partial_p \phi_\ell \, \partial_q \phi_\ell - \frac{1}{2} m^2 \phi_\ell^2 g_{jk}.$$

To obtain a physically meaningful model, the stress-energy tensor T needs to satisfy the conservation law

$$\nabla_p (g^{pj} T_{jk}) = 0, \quad k = 1, 2, 3, 4.$$

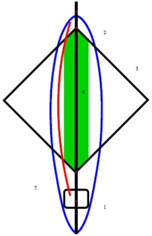
Let $V_{\widehat{g}} \subset M$ be a neighborhood of the geodesic μ and $p^-, p^+ \in \mu$.

Theorem (Kurylev-Lassas-Oksanen-U, 2022; U-Wang, 2020)

Let

$$\begin{aligned}\mathcal{D} \;=\; & \{(V_g, g|_{V_g}, \phi|_{V_g}, \mathcal{F}|_{V_g});\ g \textit{ and } \phi \textit{ satisfy Einstein equations} \\ & \quad \textit{with a source } \mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2), \textit{ supp } (\mathcal{F}) \subset V_g, \textit{ and} \\ & \quad \nabla_j(\mathrm{T}^{jk}(g,\phi) + \mathcal{F}_1^{jk}) = 0\}.\end{aligned}$$

The data set $\mathcal{D}$ determines uniquely the metric on the double cone $(J^+(p^-) \cap J^-(p^+), \widehat{g})$.



Inverse Boundary Value Problem

Assume $M = \mathbb{R} \times N$ is a Lorentzian manifold of dimension $(1+3)$ with time-like boundary.

$$\begin{aligned} \Box_g u(x) + a(x)u(x)^4 &= 0, && \text{on } M, \\ u(x) &= f(x), && \text{on } \partial M, \\ u(t,y) &= 0, && t < 0, \end{aligned}$$

Inverse Problem: determine the metric g and the coefficient a from the Dirichlet-to-Neumann map.

The Main Result

Theorem (Hintz-U-Zhai, 2022)

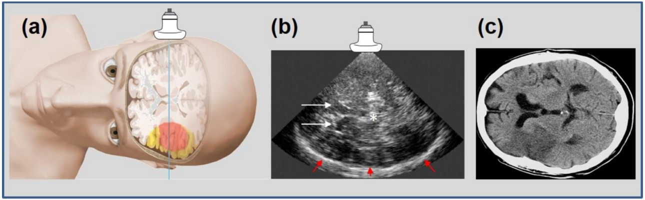
Consider the semilinear wave equations

$$\Box_{g^{(j)}} u(x) + a^{(j)} u(x)^4 = 0, \qquad j = 1, 2,$$

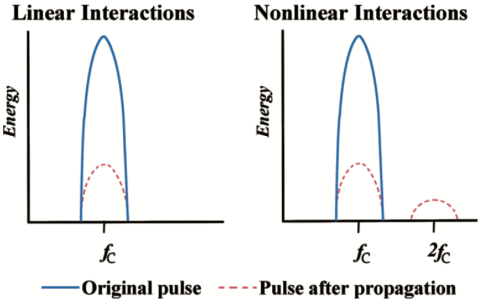
on Lorentzian manifold $M^{(j)}$ with the same boundary $\mathbb{R} \times \partial N$. If the Dirichlet-to-Neumann maps $\Lambda^{(j)}$ acting on $\mathcal{C}^5([0,T] \times \partial N)$ are equal, $\Lambda^{(1)} = \Lambda^{(2)}$, then there exist a diffeomorphism $\Psi \colon U_{g^{(1)}} \to U_{g^{(2)}}$ with $\Psi|_{(0,T)\times\partial N} = \mathcal{I}d$ and a smooth function $\beta \in C^\infty(M^{(1)})$, $\beta|_{(0,T)\times\partial N} = \partial_\nu \beta|_{(0,T)\times\partial N} = 0$, so that, in $U_{g^{(1)}}$,

$$\Psi^* g^{(2)} = e^{-2\beta} g^{(1)}, \quad \Psi^* a^{(2)} = e^{-\beta} a^{(1)}, \quad \Box_g e^{-\beta} = 0.$$

Ultrasound Imaging



Nonlinear interaction: waves at frequency f_C generate waves at frequency $2f_C$:



Inverse Boundary Value Problem

The acoustic waves are modeled by the Westervelt-type equation

$$\begin{aligned}\frac{1}{c^2(x)}\partial_t^2 p(t,x) - \beta(x)\partial_t^2 p^2(t,x) &= \Delta p(t,x), \quad \text{in } (0,T)\times\Omega,\\ p(t,x) &= f, \quad \text{on } (0,T)\times\partial\Omega,\\ p = \frac{\partial p}{\partial t} &= 0, \quad \text{on } \{t=0\},\end{aligned}$$

- c: wavespeed
- β: nonlinear parameter

Inverse problem: recover β from the Dirichlet-to-Neumann map Λ.

Second Order Linearization

Second order linearization and the resulted integral identity:

$$\int_0^T \int_{\partial\Omega} \frac{\partial^2}{\partial\epsilon_1 \partial\epsilon_2} \Lambda(\epsilon_1 f_1 + \epsilon_2 f_2)\Big|_{\epsilon_1=\epsilon_2=0} f_0 \, dS \, dt$$
$$= 2 \int_0^T \int_\Omega \beta(x) \partial_t(u_1 u_2) \partial_t u_0 \, dx \, dt.$$

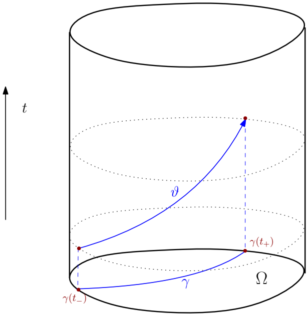
where u_j, $j = 1, 2$ are solutions to the linear wave equation

$$\frac{1}{c^2} \partial_t^2 u_i(t, x) - \Delta u_j(t, x) = 0$$

with $u_j|_{(0,T)\times\partial\Omega} = f_j$, and u_0 is the solution to the backward wave equation with $u_0|_{(0,T)\times\partial\Omega} = f_0$

Reduction to a Weighted Ray Transform

Construct Gaussian beam solutions u_0, u_1, u_2 traveling along the same null-geodesic $\vartheta(t) = (t, \gamma(t))$, where $\gamma(t), t \in (t_-, t_+)$ is the geodesic in (Ω, g) joining two boundary points $\gamma(t_-), \gamma(t_+) \in \partial\Omega$.



Insert into the integral identity, one can extract the Jacobi-weighted ray transform of $f = \beta c^{3/2}$ $\Rightarrow$ invert this weighted ray transform (Paternain-Salo-U-Zhou, 2019; Feizmohammadi-Oksanen, 2020)

A frequency-domain model (with constant wavespeed $c \equiv 1$): two plane waves $e^{ik\cdot x}$ at frequency $\omega = |k|$ generate nonlinear wave $\mathcal{V}$ at frequency 2ω:

$$\Delta \mathcal{V} + (2\omega)^2 \mathcal{V} = \beta e^{2ik\cdot x},$$

which can be factorized into

$$\left(\partial_x - \Lambda^+_{2\omega}\right)\left(\partial_x - \Lambda^-_{2\omega}\right)\mathcal{V} = \beta e^{2ik\cdot x}$$

where $\Lambda^+_{2\omega}$ is the forward DtN map. In 2D:

$$\Lambda^+_{2\omega} V = 2i\omega V - \frac{1}{4i\omega}\partial_y^2 V + \mathcal{O}(1/\omega^2).$$

Numerical results for recovering β in the following equation from $V|_{\partial\Omega}, \partial_\nu V|_{\partial\Omega}$

$$\partial_x V - 2i\omega V + \frac{1}{4i\omega}\partial_y^2 V = \beta e^{2ik\cdot x}.$$

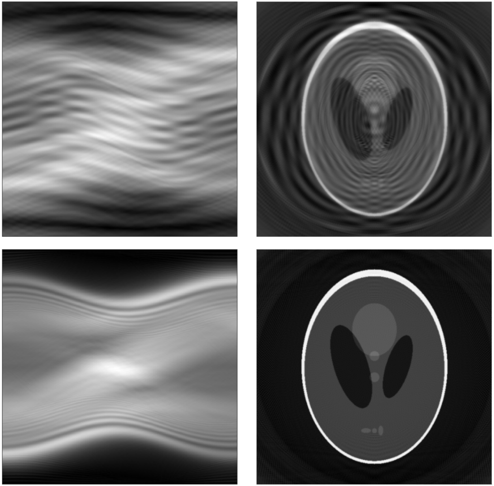


Figure: $L/\lambda = 10$ (top row) and $L/\lambda = 100$ (bottom row) where L is the size of the image and λ is the wavelength.

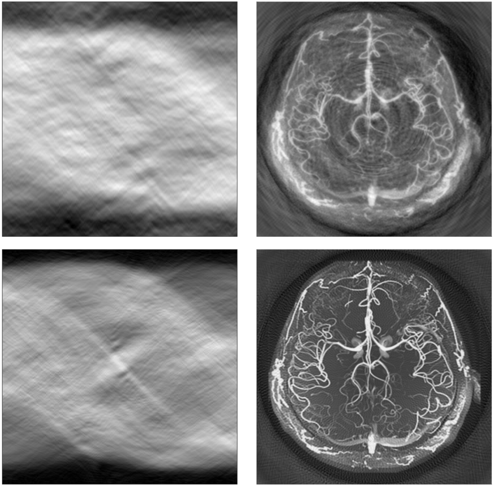


Figure: $L/\lambda = 10$ (top row) and $L/\lambda = 100$ (bottom row) where L is the size of the image and λ is the wavelength.

Other Developments

1. Einstein's equations (U-Wang, 2020)
2. Non-linear elasticity (de Hoop-U-Wang, 2020; U-Zhai, 2021)
3. Yang-Mills (Chen-Lassas-Oksanen-Paternain, 2021, 2022)
4. Inverse Scattering (Sa Barreto-U-Wang, 2022; Hintz-Sa Barreto-U-Zhang, 2024)
5. Semilinear equations (Kurylev-Lassas-U, 2018; Wang-Zhou, 2019; Hintz-U-Zhai, 2020; Stefanov-Sa Barreto, 2021, 2022)
6. Non-linear Acoustics (Eptaminitakis-Stefanov, 2022)
7. Non-linear Dirac (Yi, 2024)

Fractional Laplacian

Consider the *fractional Laplacian*

$$(-\Delta)^s, \quad 0 < s < 1,$$

defined via the Fourier transform by

$$(-\Delta)^s u = \mathcal{F}^{-1}\{|\xi|^{2s}\widehat{u}(\xi)\}.$$

This operator is *nonlocal*: it does not preserve supports, and computing $(-\Delta)^s u(x)$ involves values of u far away from x.

Fractional Laplacian

Different models for diffusion:

$\partial_t u - \Delta u = 0$	normal diffusion/BM
$\partial_t u + (-\Delta)^s u = 0$	superdiffusion/Lévy flight
$\partial_t^\alpha u - \Delta u = 0$	subdiffusion/CTRW

The *fractional Laplacian* is related to

- anomalous diffusion involving long range interactions (turbulent media, population dynamics)
- Lévy processes in probability theory
- financial modelling with jump processes

Many results for time-fractional inverse problems [Kaltenbacher-Rundell, 2023].

Inverse Problem for the Fractional Laplacian

Let $\Omega \subset \mathbb{R}^n$ bounded, $q \in L^\infty(\Omega)$. Since $(-\Delta)^s$ is nonlocal, the Dirichlet problem becomes

$$\begin{cases} ((-\Delta)^s + q)u = 0 & \text{in } \Omega, \\ u = f & \text{in } \Omega_e \end{cases}$$

where $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$ is the *exterior domain*.
Given $f \in H^s(\Omega_e)$, look for a solution $u \in H^s(\mathbb{R}^n)$. DN map

$$\Lambda_{q,s} : H^s(\Omega_e) \to H^{-s}(\Omega_e), \quad \Lambda_{q,s} f = (-\Delta)^s u|_{\Omega_e}.$$

Inverse problem: given $\Lambda_{q,s}$, determine q.

First Result

Theorem(Ghosh–Salo–U, 2020)
Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $0 < s < 1$, and let $q_1, q_2 \in L^\infty(\Omega)$. If $W, W' \subset \Omega_e$ are open sets, and if

$$\Lambda_{q_1,s} f|_{W'} = \Lambda_{q_2,s} f|_{W'}, \quad f \in C_c^\infty(W),$$

then $q_1 = q_2$ in Ω.
Remark: Only one f is enough (Ghosh-Rüland-Salo-U, 2018)
Main features:

- local data result for *arbitrary* $W, W' \subset \Omega_e$
- the same method works for *all* $n \geq 2$
- new mechanism for solving (nonlocal) inverse problems

Identity

$\Lambda_{q_1,s} = \Lambda_{q_2,s}$ implies

$$\boxed{\int_\Omega (q_1 - q_2) u_1 u_2 = 0}$$

where u_i satisfies

$$((-\Delta)^s + q_i)u_i = 0 \quad \text{on } \Omega$$

Instead of CGO solutions we will use Runge approximation for non-local operators.

Runge approximation

Classical Runge property (for $\bar{\partial} u = 0$): analytic functions in simply connected $U \subset \mathbb{C}$ can be approximated by complex polynomials.

General Runge property (for elliptic PDE): any solution in U, where $U \subset \Omega \subset \mathbb{R}^n$ can be approximated using solutions in Ω.

Reduces by duality to the unique continuation principle (Lax–Malgrange, 1956) cf. approximate controllability.

Runge approximation

Produce solutions with $u|_{U_0} \approx 0$ and $u|_{U_1} \gg 1$ (region of interest), but with very little control outside $U_0 \cup U_1$. Useful in the Calderón problem for

- boundary determination (Kohn–Vogelius 1984)
- piecewise analytic conductivities (Kohn-Vogelius 1985)
- local data if γ is known near $\partial\Omega$ (Ammari-U 2004)
- detecting shapes of obstacles (γ known near $\partial\Omega$), e.g. singular solutions (Isakov 1988); probe method (Ikehata 1998); oscillating-decaying solutions (Nakamura-U-Wang 2005); monotonicity tests (Harrach 2008)

Main tools 1: Uniqueness

Theorem
If $u \in H^{-r}(\mathbb{R}^n)$ for some $r \in \mathbb{R}$, and if $u|_W = (-\Delta)^s u|_W = 0$ for some open set $W \subset \mathbb{R}^n$, then $u \equiv 0$.
Proof (sketch). If u is nice enough, then

$$(-\Delta)^s u \sim \lim_{y \to 0} y^{1-2s} \partial_y w(\,\cdot\,, y)$$

where $w(x, y)$ is the *Caffarelli-Silvestre extension* of u:

$$\begin{cases} \mathit{div}_{x,y}(y^{1-2s} \nabla_{x,y} w) = 0 & \text{in } \mathbb{R}^n \times \{y > 0\}, \\ w|_{y=0} = u. & \end{cases}$$

Thus $(-\Delta)^s u$ is obtained from a *local equation*, which is degenerate elliptic with A_2 weight y^{1-2s}. Carleman estimates [Rüland 2015] and $u|_W = (-\Delta)^s u|_W = 0$ imply uniqueness.

Caffarelli-Silvestre Extension

$$\begin{cases} \Delta w = 0 & \text{in } \mathbb{R}^n \times \{y > 0\}, \\ w\big|_{y=0} = u. \end{cases}$$

$$w(x,y) = \int e^{ix\cdot\xi} e^{-y|\xi|} \widehat{u}(\xi) d\xi$$

$$\Lambda u(x) = -\lim_{y\to 0} \partial_y w(\cdot, y) = (-\Delta)^{\frac{1}{2}}(u)(x)$$

Main tools 2: Approximation

Theorem(Ghosh–Salo–U, 2020)
Any $f \in L^2(\Omega)$ can be approximated in $L^2(\Omega)$ by *solutions* $u|_\Omega$, where

$$((-\Delta)^s + q)u = 0 \text{ in } \Omega, \qquad supp(u) \subset \overline{\Omega} \cup \overline{W}. \qquad (*)$$

If everything is C^∞, any $f \in C^k(\overline{\Omega})$ can be approximated in $C^k(\overline{\Omega})$ by functions $d(x)^{-s}u|_\Omega$ with u as in $(*)$.
Proof. Apply this to

$$\int_\Omega (q_1 - q_2)u_1 u_2 = 0, \quad (-\Delta)^s + q_j)u_j = 0 \, (j = 1, 2)$$

Anisotropic Case

$\gamma(x) = (\gamma^{ij}(x))$, (γ^{ij}) positive definite on Ω, $(\gamma^{ij}) = Id$ on Ω_e.

$$\begin{cases} \left(\left(\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\gamma^{ij}\frac{\partial}{\partial x_j}\right)\right)^s + q\right)u = 0 & \text{in } \Omega, \\ u = f & \text{in } \Omega_e \end{cases}$$

where $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$ is the *exterior domain*.

$$\Lambda_{q,s,\gamma} f = \mathcal{L}^s(u)|_{\Omega_e},$$

Theorem(Ghosh–Lin–Xiao, 2017) Can determine uniquely q from $\Lambda_{q,s,\gamma}$
Nonlocality helps!

Fractional Laplacian for Variable Coefficients

Definition of $\mathcal{L}^s := \left(\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(\gamma^{ij}\frac{\partial}{\partial x_i}\right)\right)^s$ via heat semi-group $\{e^{-t\mathcal{L}}\}_{t\geq 0}$:

$$\forall x \in \mathbb{R}^n, \quad \mathcal{L}^s u(x) := \frac{1}{\Gamma(-s)} \int_0^\infty \frac{U(x,t) - u(x)}{t^{1+s}}\, dt,$$

where U uniquely solves

$$\begin{cases} \partial_t U - \mathcal{L} U = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ U\big|_{t=0} = u & \text{in } \mathbb{R}^n. \end{cases}$$

Fractional Calderón Anisotropic Problem (second instance)

Let (M, g) be a smooth closed connected Riemannian manifold of dimension $n \geq 2$. Let $-\Delta_g$ be the positive Laplace–Beltrami operator on M. It is a self-adjoint operator on $L^2(M)$ with the domain $\mathcal{D}(-\Delta_g) = H^2(M)$.
Let $\alpha \in (0, 1)$. By the functional calculus we define the fractional Laplacian $(-\Delta_g)^\alpha$ as an unbounded self-adjoint operator on $L^2(M)$ given by

$$(-\Delta_g)^\alpha u = \sum_{k=0}^{\infty} \lambda_k^\alpha \pi_k u,$$

equipped with the domain $\mathcal{D}((-\Delta_g)^\alpha) = H^{2\alpha}(M)$. Here $0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$ are the distinct eigenvalues of $-\Delta_g$ and $\pi_k : L^2(M) \to \mathrm{Ker}(-\Delta_g - \lambda_k)$ is the orthogonal projection onto the eigenspace of λ_k.

The Inverse Problem

Let $\mathcal{O} \subset M$ be open nonempty and let $f \in C_0^\infty(\mathcal{O})$ be such that $(f, 1)_{L^2(M)} = 0$. Then the equation

$$(-\Delta_g)^\alpha u = f \quad \text{in} \quad M$$

has a unique solution $u = u^f \in C^\infty(M)$ with the property that $(u^f, 1)_{L^2(M)} = 0$, given by

$$u^f = (-\Delta_g)^{-\alpha} f = \sum_{k=1}^{\infty} \lambda_k^{-\alpha} \pi_k f.$$

We define the local source–to–solution map $L_{M,g,\mathcal{O}}$ by

$$L_{M,g,\mathcal{O}}(f) := u^f|_{\mathcal{O}} = ((-\Delta_g)^{-\alpha} f)|_{\mathcal{O}}.$$

The fractional anisotropic Calderón problem: does the knowledge of $L_{M,g,\mathcal{O}}$, the observation set $\mathcal{O}$, and the metric $g|_{\mathcal{O}}$ determine the manifold (M, g) globally?

Obstruction to uniqueness: if $\Phi : M_1 \to M_2$ is a C^∞ diffeomorphism such that $\Phi^* g_2 = g_1$ on M_1 and $\Phi|_{\mathcal{O}} = \mathrm{Id}$, then $L_{M_2,g_2,\mathcal{O}} = L_{M_1,g_1,\mathcal{O}}$.

Theorem (Feizmohammadi–Ghosh–Krupchyk–U., 2021, Rülland, 2023)

Let (M_1, g_1) and (M_2, g_2) be smooth closed connected Riemannian manifolds of dimension $n \geq 2$, and let $\mathcal{O}_j \subset M_j$, $j = 1, 2$, be open nonempty connected sets. Assume that

$$(\mathcal{O}_1, g_1|_{\mathcal{O}_1}) = (\mathcal{O}_2, g_2|_{\mathcal{O}_2}) := (\mathcal{O}, g).$$

Assume furthermore that

$$L_{M_2,g_2,\mathcal{O}}(f) = L_{M_1,g_1,\mathcal{O}}(f),$$

for all $f \in C_0^\infty(\mathcal{O})$ with $(f, 1)_{L^2(\mathcal{O})} = 0$. Then there exists a diffeomorphism $\Phi : M_1 \to M_2$ such that $\Phi^* g_2 = g_1$ on M_1, and $\Phi|_{\mathcal{O}} = \mathrm{Id}$.

Remarks

Remark. While the anisotropic Calderón problem is wide open in dimensions $n \geq 3$, here we are able to recover a smooth closed Riemannian manifold, up to a natural obstruction.

Remark. While there is an additional obstruction in the geometric version of the anisotropic Calderón problem in dimension $n = 2$, coming from the conformal invariance of the Laplacian, this obstruction is not present in the fractional anisotropic Calderón problem.

Sketch of the proof

- Step 1. Pass from the equality

$$L_{M_2,g_2,\mathcal{O}}(f) = L_{M_1,g_1,\mathcal{O}}(f),$$

for all $f \in C_0^\infty(\mathcal{O})$ with $(f,1)_{L^2(\mathcal{O})} = 0$, to the equality for the heat kernels of $P_{g_j} = -\Delta_{g_j}$ on $\mathcal{O}$,

$$e^{-tP_{g_1}}(x,y) = e^{-tP_{g_2}}(x,y), \quad x,y \in \mathcal{O}, \quad t > 0.$$

- Step 2. Show that the equality

$$e^{-tP_{g_1}}(x,y) = e^{-tP_{g_2}}(x,y), \quad x,y \in \mathcal{O}, \quad t > 0.$$

implies that there exists a diffeomorphism $\Phi : M_1 \to M_2$ such that $\Phi^* g_2 = g_1$ on M_1, and $\Phi|_{\mathcal{O}} = \mathrm{Id}$.

Sketch of the proof of Step 1

The key role is played by the following representation of the operator $P_{g_j}^{-\alpha} = (-\Delta_{g_j})^{-\alpha}$ in terms of the heat semigroup $e^{-tP_{g_j}}$,

$$P_{g_j}^{-\alpha} v_j = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-tP_{g_j}} v_j \frac{1}{t^{1-\alpha}} dt,$$

where $v_j \in L^2(M_j)$ is such that $(v_j, 1)_{L^2(M_j)} = 0$, $j = 1, 2$.
Let $f \in C_0^\infty(\mathcal{O})$. As $g_1|_{\mathcal{O}} = g_2|_{\mathcal{O}} = g$ and $(\Delta_g^m f, 1)_{L^2(\mathcal{O})} = 0$, we get from the equality of the local source–to–solution maps that

$$L_{M_2,g_2,\mathcal{O}}(\Delta_g^m f) = L_{M_1,g_1,\mathcal{O}}(\Delta_g^m f), \;\; m = 1, 2, \dots .$$

Therefore,

$$\left(P_{g_1}^{-\alpha}\Delta_g^m f\right)|_{\mathcal{O}} = \left(P_{g_2}^{-\alpha}\Delta_g^m f\right)|_{\mathcal{O}}.$$

Hence,

$$\int_0^\infty \left(\left(e^{-tP_{g_1}} - e^{-tP_{g_2}}\right)\Delta_g^m f\right)(x)\frac{dt}{t^{1-\alpha}} = 0, \quad x \in \mathcal{O},$$

for all $m = 1, 2, \ldots$.
Using that

$$\left(e^{-tP_{g_j}}\Delta_g^m f\right)(x) = \partial_t^m\left(e^{-tP_{g_j}}f\right)(x), \quad x \in \mathcal{O},$$

we get

$$\int_0^\infty \partial_t^m\left(\left(e^{-tP_{g_1}} - e^{-tP_{g_2}}\right)f\right)(x)\frac{dt}{t^{1-\alpha}} = 0, \quad x \in \mathcal{O},$$

for all $m = 1, 2, \ldots$.

Next, we would like to integrate by parts m times. In doing so, we let $\omega_1 \subset\subset \mathcal{O}$ be open nonempty, and pick $\omega_2 \subset\subset \mathcal{O}$ such that $\overline{\omega_1} \cap \overline{\omega_2} = \emptyset$. We work with $f \in C_0^\infty(\omega_1)$, and restrict $x \in \omega_2$. No contributions coming from the end points when integrating by parts will occur, thanks to the pointwise upper Gaussian estimate on the heat kernel,

$$|e^{-tP_{g_j}}(x,y)| \le Ct^{-\frac{n}{2}} e^{-\frac{c d_{g_j}(x,y)^2}{t}}, \quad 0 < t < 1, \quad x, y \in M_j,$$

together with the estimate

$$\|e^{-tP_{g_j}} v\|_{L^2(M_j)} \le e^{-\beta t} \|v\|_{L^2(M_j)}, \quad t \ge 0,$$

for some $\beta > 0$, $v \in L^2(M_j)$, $(v,1)_{L^2(M_j)} = 0$.

Thus, integrating by parts m times, we obtain that

$$\int_0^\infty \big((e^{-tP_{g_1}} - e^{-tP_{g_2}})f\big)(x)\frac{dt}{t^{1+m-\alpha}} = 0, \quad x \in \omega_2,$$

for all $m = 1, 2, \dots$. Making the change of variables $s = 1/t$, we get

$$\int_0^\infty \varphi(s)s^m ds = 0, \quad m = 0, 1, 2, \dots,$$

where

$$\varphi(s) = \frac{\big((e^{-\frac{1}{s}P_{g_1}} - e^{-\frac{1}{s}P_{g_2}})f\big)(x)}{s^\alpha}, \quad x \in \omega_2.$$

Standard complex analytic arguments show that

$$\big((e^{-tP_{g_1}} - e^{-tP_{g_2}})f\big)(x) = 0, \quad x \in \omega_2, \quad t > 0.$$

By the unique continuation for the heat equation,

$$e^{-tP_{g_1}}(x, y) = e^{-tP_{g_2}}(x, y), \quad x, y \in \mathcal{O}, \quad t > 0.$$

Sketch of the proof of Step 2

We shall reduce the problem to an inverse problem for the wave equation with interior measurements on $\mathcal{O}$.
To that end, let $F \in C_0^\infty((0,\infty) \times \mathcal{O})$ and consider the following inhomogeneous initial value problem for the wave equation,

$$\begin{cases} (\partial_t^2 - \Delta_{g_j}) u_j(t,x) = F(t,x), & (t,x) \in (0,\infty) \times M_j, \\ u_j(0,x) = 0, & x \in M_j, \\ \partial_t u_j(0,x) = 0, & x \in M_j, \end{cases}$$

$j = 1, 2$. The problem has a unique solution
$u_j = u_j^F \in C^\infty([0,\infty) \times M_j)$. We define the local source–to–solution map on $\mathcal{O}$ by

$$L^{\text{wave}}_{M_j, g_j, \mathcal{O}} : F \mapsto u_j^F|_{\mathcal{O}}.$$

Our goal is to show that the equality for the heat kernels

$$e^{-tP_{g_1}}(x,y) = e^{-tP_{g_2}}(x,y), \quad x, y \in \mathcal{O}, \quad t > 0.$$

implies the equality of the local source–to–solution maps,

$$L^{\text{wave}}_{M_1,g_1,\mathcal{O}}(F) = L^{\text{wave}}_{M_2,g_2,\mathcal{O}}(F),$$

for $F \in C_0^\infty((0,\infty)\times\mathcal{O})$.
The existence of a desired diffeomorphism will then follow from the boundary control method of Belishev (1987), Belishev-Kurylev (1992), which is based on the unique continuation result of Tataru (1995).
In doing so, we write for $j = 1, 2$,

$$u_j^F(t,x) = \int_0^t \frac{\sin((t-s)\sqrt{P_{g_j}})}{\sqrt{P_{g_j}}} F(s,x)ds, \ (t,x) \in [0,\infty)\times M_j.$$

Thanks to the transmutation formula of Kannai, 1977,

$$e^{-tP_{g_j}} v_j = \frac{1}{4\pi^{1/2} t^{3/2}} \int_0^{+\infty} e^{-\frac{\tau}{4t}} \frac{\sin(\sqrt{\tau}\sqrt{P_{g_j}})}{\sqrt{P_{g_j}}} v_j d\tau, \quad t > 0,$$

where $v_j \in L^2(M_j)$, $j = 1, 2$, which transforms the solution to the wave equation to the solution of the heat equation, the equality of the heat kernels implies that

$$\int_0^{+\infty} e^{-\tau t} \left(\frac{\sin(\sqrt{\tau}\sqrt{P_{g_1}})}{\sqrt{P_{g_1}}} f \right)(x) d\tau \\ = \int_0^{+\infty} e^{-\tau t} \left(\frac{\sin(\sqrt{\tau}\sqrt{P_{g_2}})}{\sqrt{P_{g_2}}} f \right)(x) d\tau, \ x \in \mathcal{O}, \ t > 0,$$

for $f \in C_0^\infty(\mathcal{O})$. Inverting the Laplace transform, we get

$$u_1^F(t, x) = u_2^F(t, x), \quad x \in \mathcal{O}, \quad t > 0,$$

showing the claim.

Further Results

- Regularity and stability (Rüland–Salo, 2017)
- Reconstruction with single Measurement (Ghosh–Rüland–Salo–U, 2018)
- Local perturbation of the fractional Laplacian (Cekić–Lin–Rüland, 2018; Covi–Mönkkönen–Railo–U, 2020)
- Non-local Perturbations (Bhattacharyya–Ghosh–U, 2021)
- Fractional magnetic operators (Covi, 2019; Li, 2020; Lai–Zhou, 2021)
- Fractional parabolic operators (Lai–Lin–Rüland, 2020; Li, 2021)
- Fractional elasticity (Li, 2021; Covi-de Hoop-Salo, 2022)
- Fractional Laplace-Beltrami operator on closed manifolds (Feizmohammadi-Ghosh-Krupchyk-U, 2021)
- Anisotropic fractional conductivity equation (Covi, 2022)

- Fractional Dirac operator on closed manifolds (Quan-U, 2022)
- Powers of the conductivity equation (Covi-Railo-Zimmerman, 2022).
- Fractional connection Laplacian (Chien, 2022)
- From nonlocal to local (Covi-Ghosh-Rüland-U, 2023)
- From nonlocal to local for parabolic equation (C.Lin-Y.Lin-U, 2023)
- Nonlocal porous medium equations (Y.Lin-Zimmerman, 2023)