Practical Electrical Impedance Tomography with partial data

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Waves and imaging in complex media

Paris, June 10, 2025

DTU Compute
Department of Applied Mathematics and Computer Science

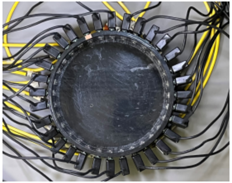
Outline

1. The Kuopio Tomography Challenge 2023
2. Level set method
3. Sparsity regularization

Goal is to share excitement about the challenge in an educational and research perspective.

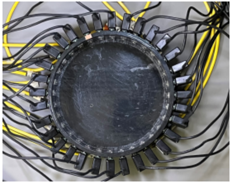
Computational **U**ncertainty **Q**uantification for **I**nverse problems

The Kuopio Tomography Challenge



[1] Räsänen et al. 2024.

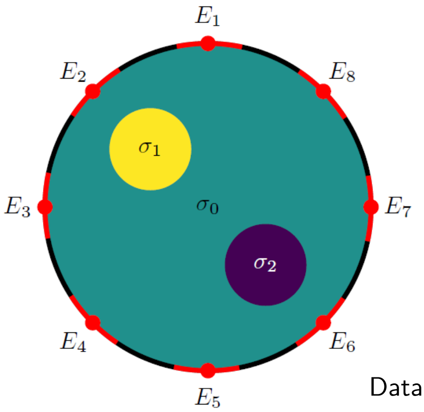
The Kuopio Tomography Challenge



- Competition organized by UEF and FIPS in 2023[1]
- 3D printed inclusions of higher and lower conductivity
- Immersed in EIT tank with 32 electrodes
- Training data available incl. empty tank data
- Seven levels of difficulty with partial data
- Input: Electrode data measuring currents and voltages
  Output: Image segmented into background/higher/lower conductivity
- Score reconstructions by SSIM on test data

[1] Räsänen et al. 2024.

The Complete Electrode Model



$$
\begin{aligned}
-\nabla \cdot (\sigma \nabla u) &= 0, && \text{in} \quad \Omega, \\
\int_{E_m} \nu \cdot (\sigma \nabla u) dS &= I_m, && \text{on} \quad E_m,\ m = 1, \ldots, M, \\
\frac{\partial u}{\partial \nu} &= 0, && \text{on} \quad \partial\Omega \setminus \bigcup_{m=1}^{M} E_m,
\end{aligned}
$$

with constraints

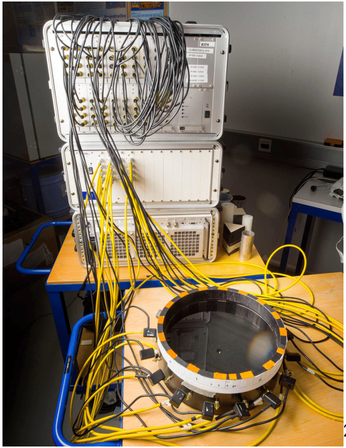
$$
u + z\sigma \frac{\partial u}{\partial \nu} = U_m, \qquad \text{on} \quad E_m,\ m = 1, \ldots, M.
$$

Data are many such pairs of currents and voltages (I, U).

Inverse problem: Find σ from the electrode data.

[1] Somersalo, Cheney, and Isaacson 1992.

Baseline Reconstruction Method - provided by UEF


[2]

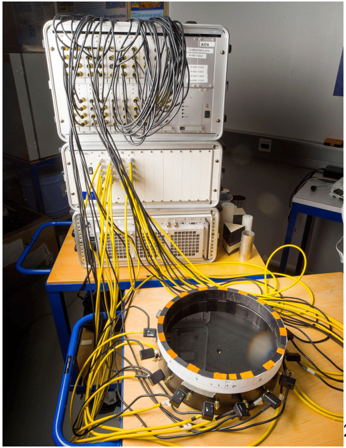
Linearization

$$\Delta V = J(\sigma_0)\Delta\sigma + \Delta e$$

with $J(\sigma_0)$ the Jacobian of the forward model.

[2] Hauptmann et al. 2017.
[3] Otsu 1979.

Baseline Reconstruction Method - provided by UEF



[2]

Linearization

$$\Delta V = J(\sigma_0)\Delta\sigma + \Delta e$$

with $J(\sigma_0)$ the Jacobian of the forward model.

MAP estimate for posterior in Bayesian approach with Gaussian assumptions

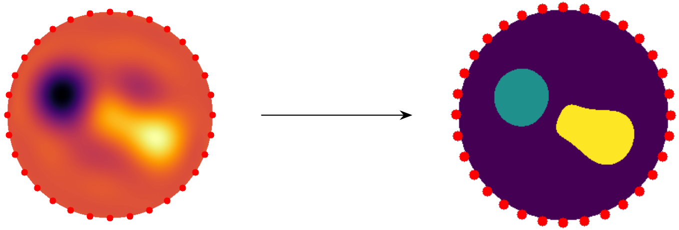
$$\underset{\Delta\sigma}{\operatorname{argmin}} \; \underbrace{\|L_{\Delta e}(J(\sigma_0)\Delta\sigma - \Delta V)\|_2^2}_{\text{Data misfit}} + \underbrace{\|L_{\text{pr}}\Delta\sigma\|_2^2}_{\text{Smoothing}};$$

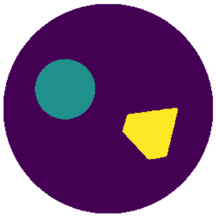
$L_{\Delta e}$ and L_{pr} are weight matrices from prior and noise covariances.

Segmentation by Otsu's method[3] on $\Delta\sigma$.

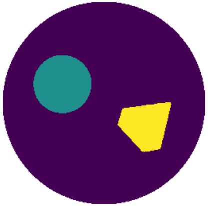
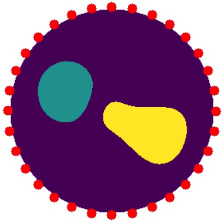
[2] Hauptmann et al. 2017.
[3] Otsu 1979.

Example: Linear Reconstruction and Segmentation

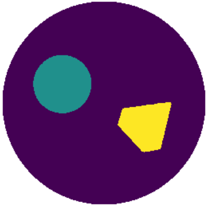
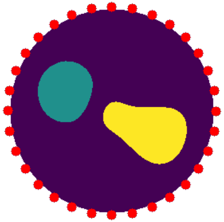
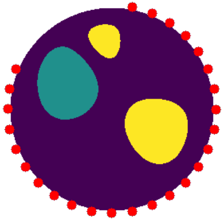




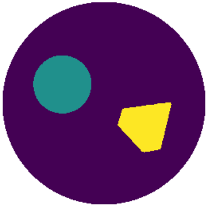
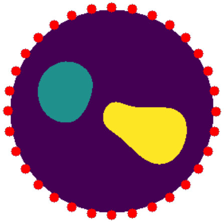
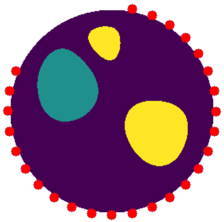
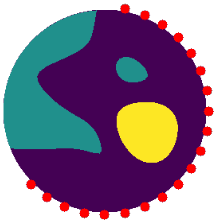
Results - Baseline Reconstruction Method

Ground Truth	Level 1
Score	0.692
	

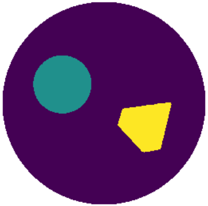
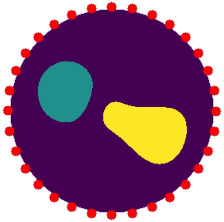
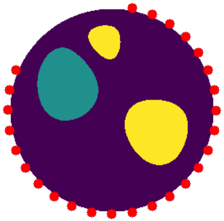
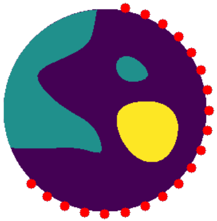
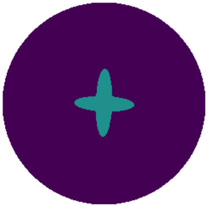
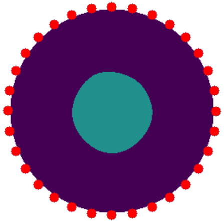
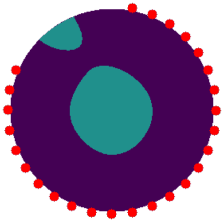
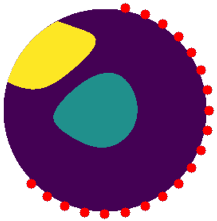
Results - Baseline Reconstruction Method

Ground Truth	Level 1	Level 4
Score	0.692	0.670
		

Results - Baseline Reconstruction Method

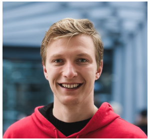
Ground Truth	Level 1	Level 4	Level 7
Score	0.692	0.670	0.355
			

Results - Baseline Reconstruction Method

Ground Truth	Level 1	Level 4	Level 7
Score	0.692	0.670	0.355
			
Score	0.615	0.585	0.134
			

The DTU-CUQI Team


Jakob Tore Kammeyer Nielsen


Martin Sæbye Carøe


Rasmus Kleist Hørlyck Sørensen


Aksel Kaastrup Rasmussen


Amal Alghamdi

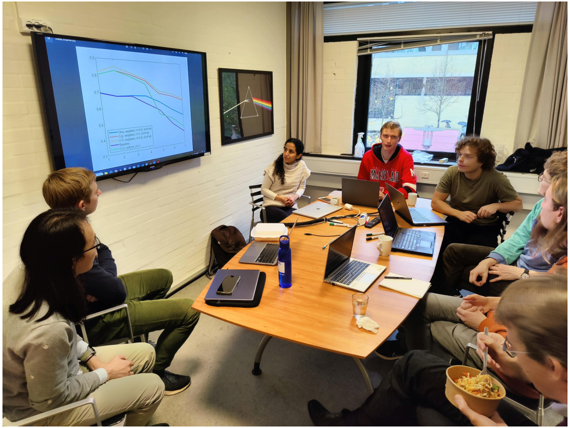

Chao Zhang


Jasper Marijn Everink


Jakob Sauer Jørgensen

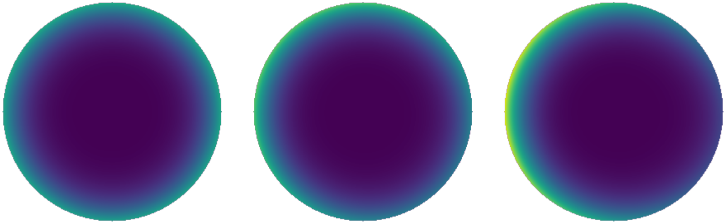

Kim Knudsen

Team spirit, group work, and coding hackathons



Spatial Regularization

Level 1	Level 4	Level 7

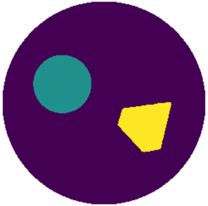
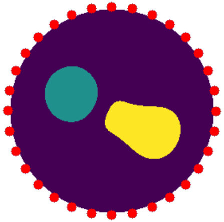
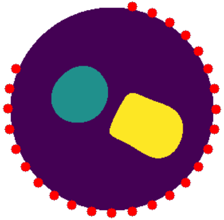
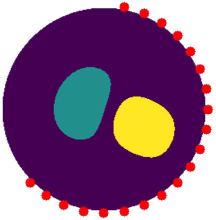
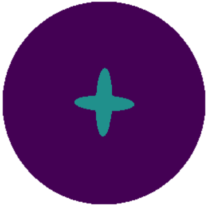
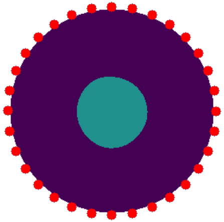
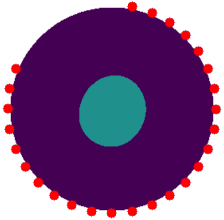
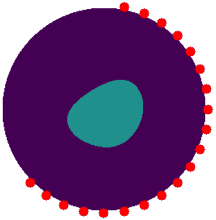


$$\underset{\Delta\sigma}{\operatorname{argmin}} \ \underbrace{\|L_{\Delta e}(J(\sigma_0)\Delta\sigma - \Delta V)\|_2^2}_{\text{Data Misfit}} + \underbrace{\|L_{\mathrm{pr}}\Delta\sigma\|_2^2}_{\text{Smoothing}} + \underbrace{\|L_{\mathrm{spatial}}\Delta\sigma\|_2^2}_{\substack{\text{Spatial}\\ \text{Regularization}}}.$$

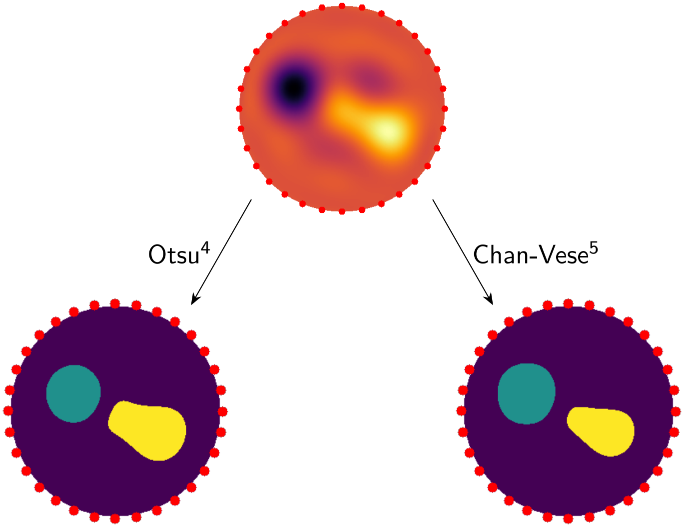
Here, L_{spatial} is a diagonal weight matrix with entries calculated proportional to

$$(L_{\mathrm{spatial}})_{i,i} = \|x_i\|_2^4 \cdot \left(\sum \mathrm{dist}(x_i, E_m)^3\right)^{\frac{1}{3}}, \quad i = 1, \ldots, N.$$

Results - Reconstruction with Spatial Regularization

Ground Truth	Level 1	Level 4	Level 7
Score	0.748	0.654	0.540
			
Score	0.665	0.678	0.663
			

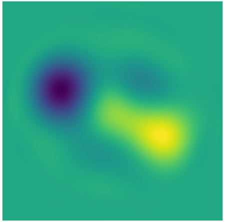
Segmentation



[4] Otsu 1979.
[5] Chan and Vese 1999.

Segmentation: Otsu's method and Chan-Vese

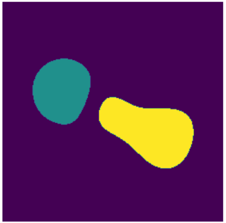
Otsu's method: Chooses two threshold values and group pixels into 3 classes such that the intra-class variance $v_{total}^2 = v_1^2 + v_2^2 + v_3^2$ is minimized. Here v_i^2 is the variance within the i'th class.



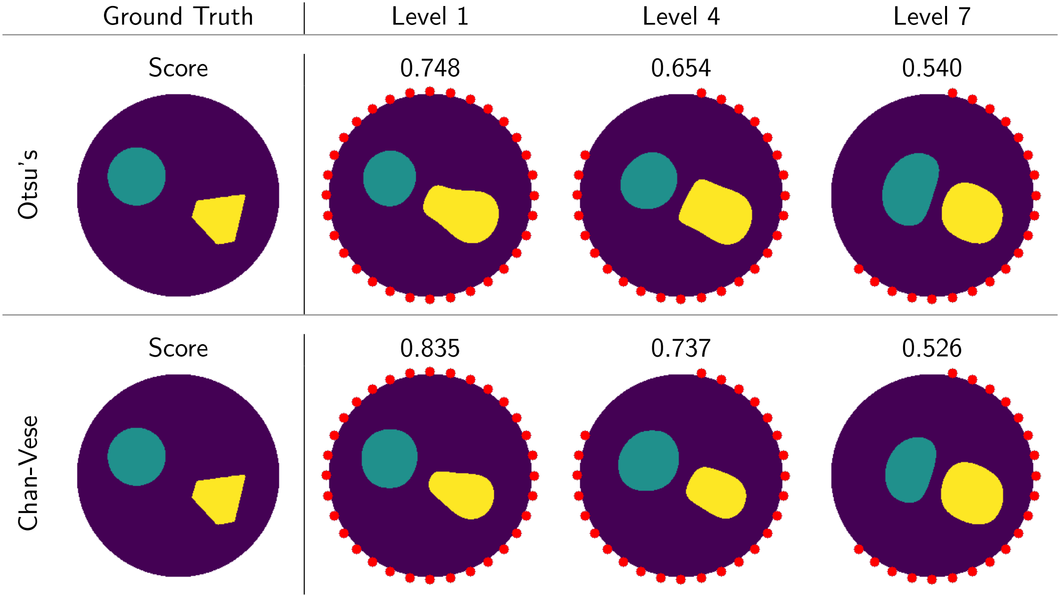
Chan-Vese: Splits the image, $u(x, y)$, into a background and inclusion by finding a curve C with closed components and $c_1, c_2 \in \mathbb{R}$ that minimize

$$F(C, c_1, c_2) = \mu \cdot \text{length}(C) + \cdot \int_{\text{inside}(C)} |u(x,y) - c_1|^2 dxdy + \int_{\text{outside}(C)} |u(x,y) - c_2|^2 dxdy,$$

where $\mu > 0$.



Results - Segmentation comparison



Level Set Parametrization

Key idea: Use smooth level-set functions ϕ_1 and ϕ_2 to describe two regions

$$\Omega_1 = \{x \in \Omega : \phi_1 > 0, \phi_2 < 0\}, \qquad \Omega_2 = \{x \in \Omega : \phi_1 < 0, \phi_2 > 0\}.$$

Parametrize conductivity from known contrasts and background $\sigma_2 < \sigma_0 < \sigma_1$

$$\sigma(\phi_1, \phi_2) = \sigma_0 \chi_{\Omega \setminus (\Omega_1 \cup \Omega_2)} + \sigma_1 \chi_{\Omega_1} + \sigma_2 \chi_{\Omega_2}. \tag{1}$$

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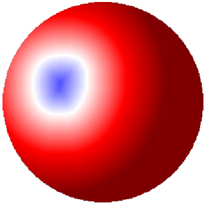
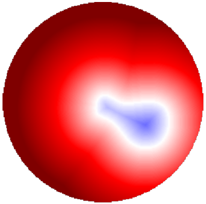
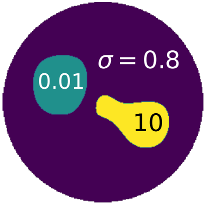


(a) ϕ_1



(b) ϕ_2



(c) σ

Level Set method

$$\underset{\sigma}{\operatorname{argmin}} \quad \|L_{\Delta e}(F(\sigma) - F(\sigma_0) - \Delta V)\|_2^2 + \beta\|\sigma\|_{\mathrm{TV}} + \|L_{\mathrm{spatial}}\sigma\|_2^2$$

[6] Chan and Tai 2004.

[7] Chung, Chan, and Tai 2005.

Level Set method

$$\underset{\phi_1,\phi_2}{\operatorname{argmin}} \quad \|L_{\Delta e}(F(\sigma) - F(\sigma_0) - \Delta V)\|_2^2 + \beta\|\sigma\|_{\mathrm{TV}} + \|L_{\mathrm{spatial}}\sigma\|_2^2$$

[6] Chan and Tai 2004.

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Level Set method

$$\underset{\phi_1,\phi_2}{\operatorname{argmin}} \quad \|L_{\Delta e}(F(\sigma) - F(\sigma_0) - \Delta V)\|_2^2 + \beta\|\sigma\|_{\mathrm{TV}} + \|L_{\mathrm{spatial}}\sigma\|_2^2$$

Algorithm[6],[7]:

1. Use previous method for a starting guess of ϕ_1, ϕ_2
2. Perform gradient descent
3. Re-initialization
4. Convergence?

[6] Chan and Tai 2004.
[7] Chung, Chan, and Tai 2005.

Step 3

Observation: When taking gradient descent steps, high-frequency components appear in ϕ_i.

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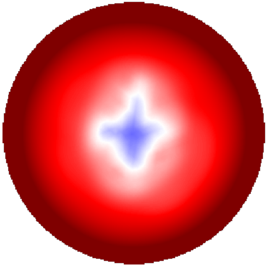
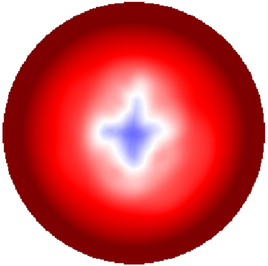


(a) Iteration 16 of ϕ_1

Step 3

Observation: When taking gradient descent steps, high-frequency components appear in ϕ_i.



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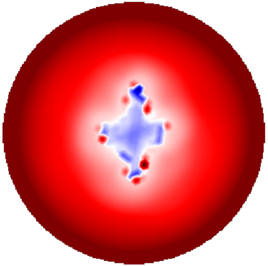
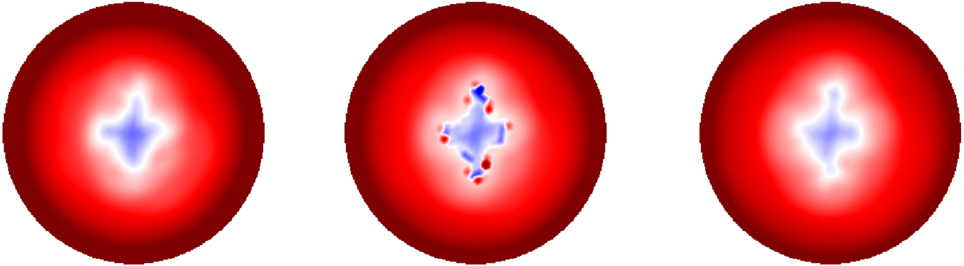


(b) Iteration 18

Step 3

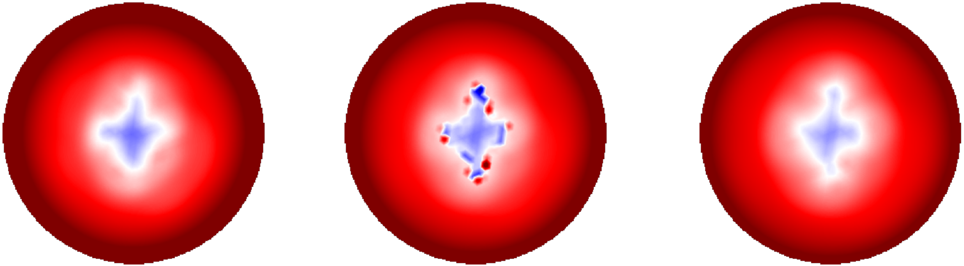
Observation: When taking gradient descent steps, high-frequency components appear in ϕ_i.



(a) Iteration 16 of ϕ_1 (b) Iteration 18 (c) Iteration 18 (after reinit.)

Step 3

Observation: When taking gradient descent steps, high-frequency components appear in ϕ_i.

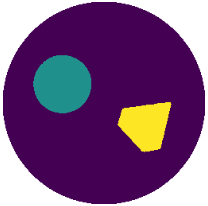
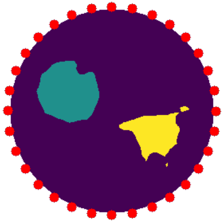
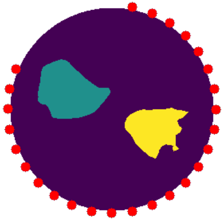
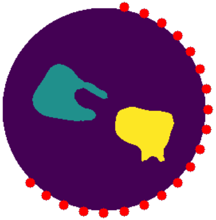
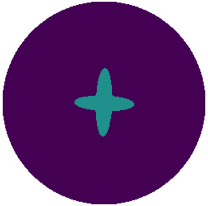
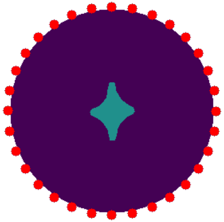
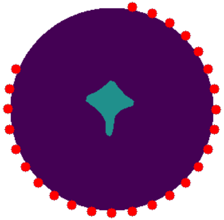
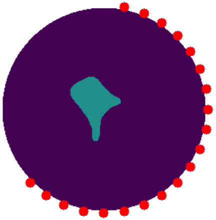


(a) Iteration 16 of ϕ_1 (b) Iteration 18 (c) Iteration 18 (after reinit.)

❸ We go from (b) to (c) (re-initialize), every now and again. This is done by solving

$$\frac{\partial \phi}{\partial t} + \mathrm{sign}(\phi)(|\nabla \phi| - 1) = 0, \quad \phi(x, 0) = \phi_i(x)$$

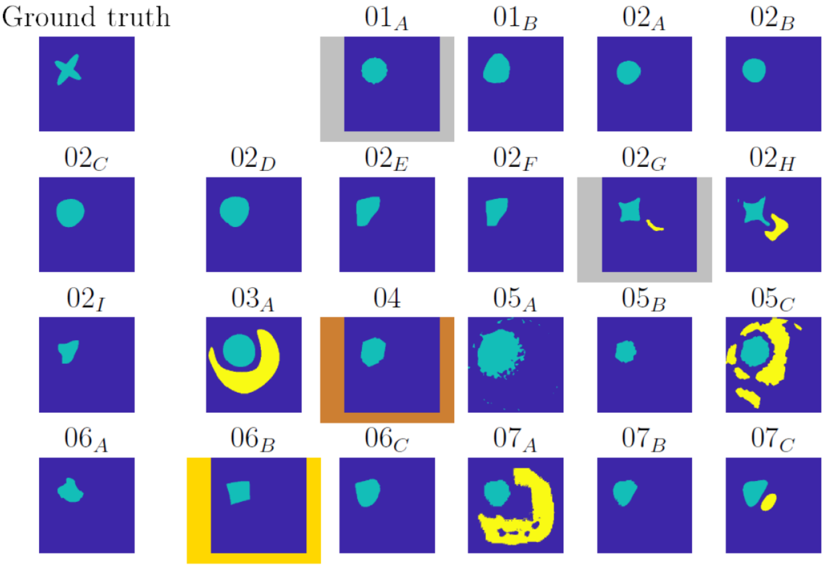
Results - Level set method

Ground Truth	Level 1	Level 4	Level 7
Score	0.881	0.866	0.798
			
Score	0.904	0.815	0.732
			

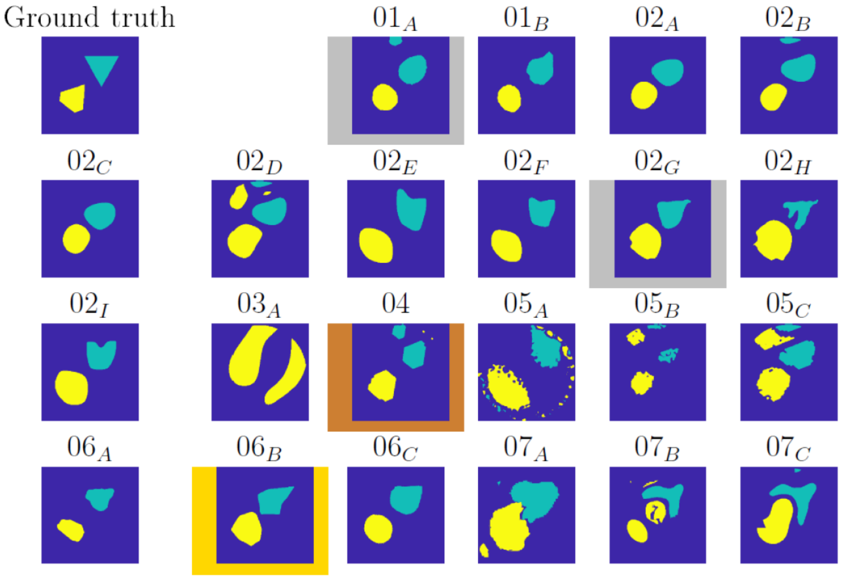
KTC 2023 participants

- 7 teams participated from Brazil, USA, Italy, France, UK, Germany, Finland, Denmark
- Several methods purely model based
- Several methods purely learning based
- Combinations of model and learning

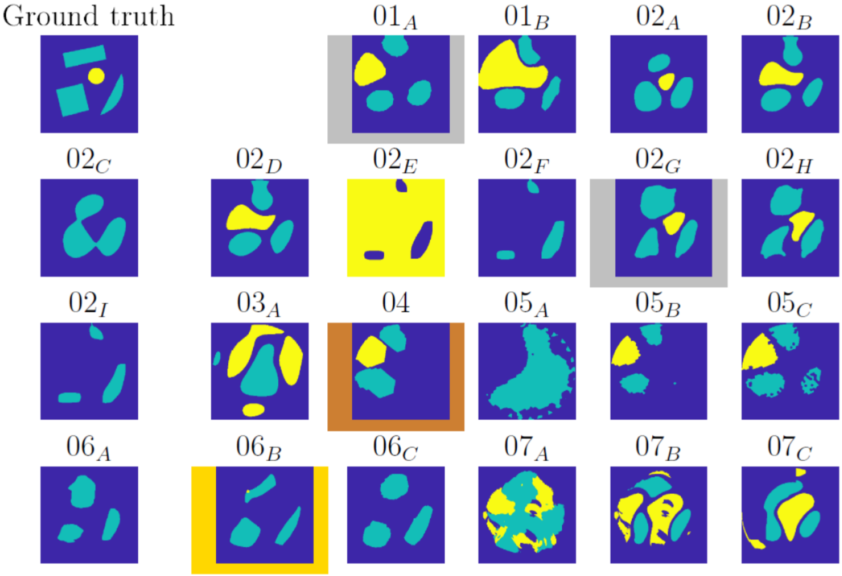
KTC 2023 results - level 1



KTC 2023 results - level 4



KTC 2023 results - level 7



Scores

Team	Level 1	Level 2	Level 3	Level 4	Level 5	Level 6	Level 7	Total score	Position
01_A	2.7332	2.49	2.096	1.7906	1.1008	1.5369	1.2217	12.9692	2nd
01_B	2.6958	2.2986	1.7392	1.337	0.89113	1.5005	1.2739	11.7361	
02_A	2.6391	2.062	1.5481	1.5187	1.1712	1.4864	1.0544	11.4798	
02_B	2.7752	2.5928	1.6347	1.6826	0.68885	1.5384	0.89256	11.8051	
02_C	2.6735	2.3062	1.539	1.5906	1.634	1.3807	1.2304	12.3545	
02_D	2.6053	2.2298	1.5614	1.34	1.0327	1.3064	1.3383	11.4139	
02_E	2.2817	1.9958	1.7565	0.74531	0.51193	0.92468	1.0003	9.2162	
02_F	2.2757	2.1358	1.1821	0.88682	0.81082	1.0623	1.1293	9.4829	
02_G	2.3189	2.5795	1.8751	1.625	1.4768	1.4757	1.6901	13.0411	2nd
02_H	2.0974	2.3789	1.7653	1.461	1.4213	1.4395	1.5434	12.1068	
02_I	2.3592	2.164	1.9833	1.3708	0.8416	1.0027	1.2678	10.9894	
03_A	1.2576	1.433	1.2946	0.5073	0.75052	0.76806	0.04045	6.0515	6th
04	2.5686	2.5025	1.7698	1.7589	1.3528	0.69271	1.1263	11.7717	3rd
05_A	1.8315	1.6339	1.6676	0.69606	0.62452	0.90694	0.88562	8.2461	
05_B	2.5278	2.1522	1.4629	1.1144	0.77772	0.26107	0.80284	9.099	
05_C	2.1233	1.6436	1.5691	1.277	1.0239	0.5796	1.0899	9.3065	4th
06_A	2.7253	2.6482	2.2765	1.8587	2.0567	1.9751	1.4228	14.9633	
06_B	2.8436	2.6952	2.6248	1.9391	2.2139	2.0089	1.703	16.0285	1st
06_C	2.7888	2.6357	2.5439	1.8836	2.0827	1.837	1.7794	15.5512	
07_A	2.2228	1.5391	1.0198	0.79026	0.9184	0.49126	0.55665	7.5383	
07_B	2.7796	2.643	1.2223	1.1925	1.1958	0.78049	0.86157	10.6753	5th
07_C	2.2373	2.4427	1.1819	1.351	0.74942	0.91403	0.87905	9.7555	

Sparsity regularization

For chosen basis $\{\phi_i\}$, consider l^1 minimization

$$\operatorname*{argmin}_x \left\{ \frac{1}{2}\|Kx - y\|^2 + \alpha \sum_i |(x, \phi_i)| \right\}.$$

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Remarkable fact: $x = \gamma\phi_j$ does *not* solve

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$$\operatorname*{argmin}_x \left\{ \frac{1}{2}\|Kx - K\phi_j\|^2 + \alpha \sum_i |(x, \phi_i)| \right\}.$$

But $x = \gamma\phi_j$ solves

$$\min_x \left\{ \frac{1}{2}\|K_k^\dagger K_k x - K_k^\dagger K\phi_j\|^2 + \alpha \sum_i w_{k,i}|(x, \phi_i)| \right\}$$

for K_k the restricted/truncated SVD operator and weights

$$w_{k,i} = \begin{cases} \|K_k^\dagger K_k \phi_i\| & \text{if } \|K_k^\dagger K_k \phi_i\| \geq \tau, \\ \tau & \text{if } \|K_k^\dagger K_k \phi_i\| < \tau. \end{cases}$$

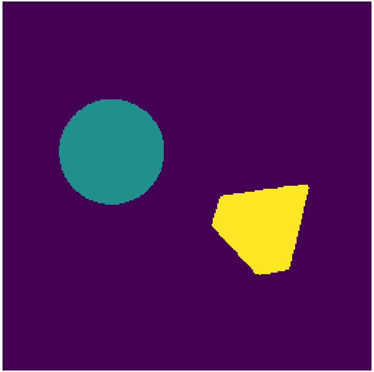
Linearized EIT

We approach the Kuopio Tomography Challenge 2023 with $K = J(x_0)$ and $\{\phi_i\}$ the computational FEM basis by weighted sparsity

$$\operatorname*{argmin}_x \left\{ \frac{1}{2} \|K_k \Delta\sigma - \Delta V\|^2 + \alpha \sum_i w_{k,i} |(\Delta\sigma, \phi_i)| \right\}$$

with K_k the restricted linearized forward operator.

Linearized EIT



(a) True inclusions

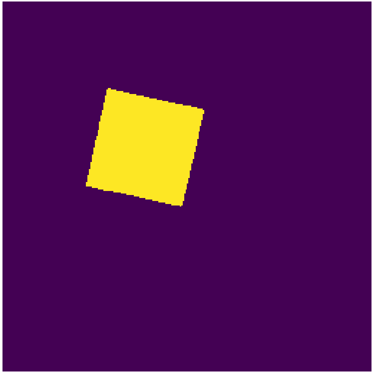
(b) Unweighted

(c) Weighted

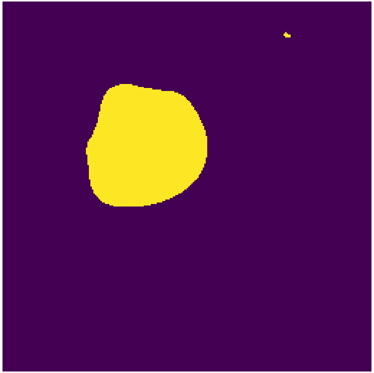
Linearized EIT

(a) True inclusion (b) Unweighted (c) Weighted

Linearized EIT


(a) True inclusion


(b) Unweighted


(c) Weighted

Final remarks

- Code provided from organizers used as base + FEniCS
- https://github.com/CUQI-DTU/KTC2023-CUQI7
- The FIPS challenges include, Deblurring (2021), CT(2022), EIT (2023) and Speech (2024). Great data sets for research and education.

References

1. Amal Mohammed A Alghamdi, Martin Sæbye Carøe, Jasper Marijn Everink, Jakob Sauer Jørgensen, Kim Knudsen, Jakob Tore Kammeyer Nielsen, Aksel Kaastrup Rasmussen, Rasmus Kleist Hørlyck Sørensen and Chao Zhang,
   Spatial regularization and level-set methods for experimental electrical impedance tomography with partial data, Applied Mathematics for Modern Challenges 2024, 10.3934/ammc.2024013.
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References

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Thank you for the attention!