Source Imaging and the Shower Curtain Effect

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The Shower Curtain Effect

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It is possible to see a person behind a shower curtain better than that person can see us.

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Wikipedia : The shower-curtain effect: the observation how nearby (relative to observer) phase front distortions of an optical wave are more severe than remote distortions of the same amplitude.

ChatGPT : The shower curtain effect when referring to ChatGPT is a metaphor used to describe a phenomenon where a large language model like ChatGPT can appear to understand a concept or topic better than it actually does, especially when it's closer to the information source

Imaging Source Behind Shower Curtain

$\hookrightarrow$ What is resolution as function of curtain's relative placement ?

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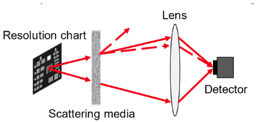
- From Alfred Hitchcock 'Psycho' (1960).

Imaging Through a Complex Section/interface

- Pei et al., Optics and lasers in Engineering., ’23;

$\rightarrow$ Imaging of *1951 USAF resolution chart* through ground glass:



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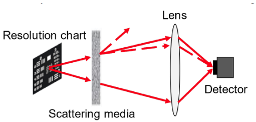

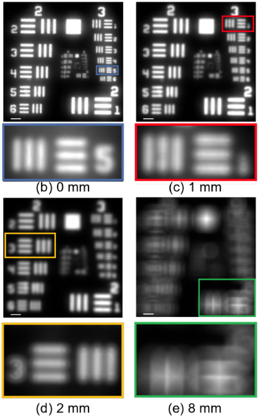

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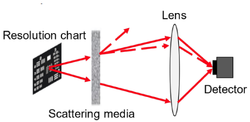
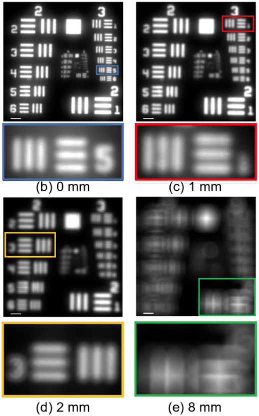
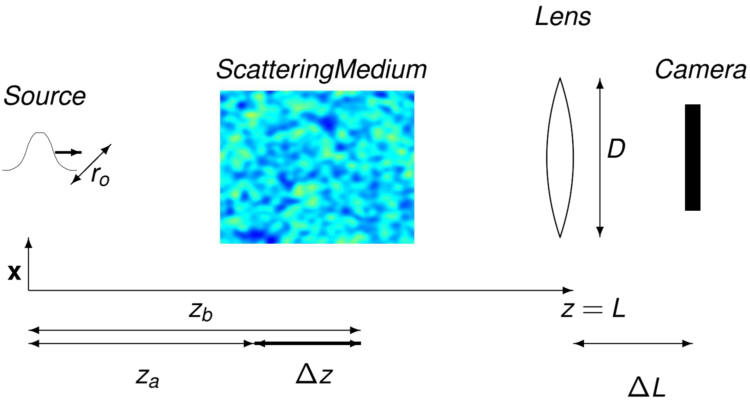




→ Empirically roughly: *‘resolution’* ∝ ‘*source curtain separation’.*

Propagation-Imaging Through Complex Sections, Set-up



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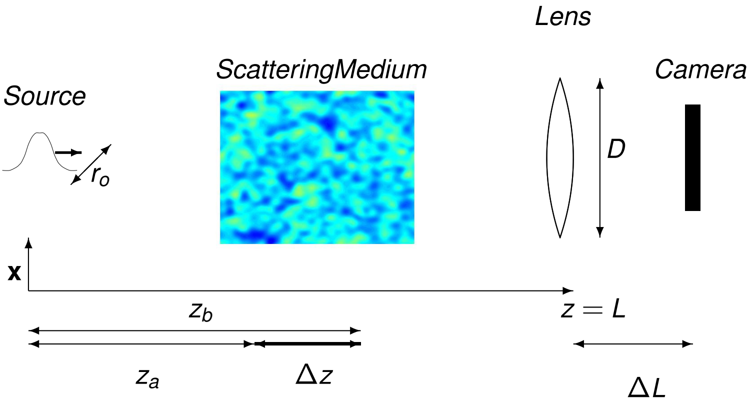

- What can one say about the signal to noise ratio ?
- How does the scattering properties of the complex section affect the shower curtain effect (quantitatively) ?
- What is the effect of bandwidth ?

Some Related Works

→ Nearfield Randomization better than farfield randomization.

• Jaruwatanadilo et al. *Optical imaging through clouds and fog*, IEEE Trans. on Geoscience and Remote Sensing '03.
• Ishimaru et al., *Time reversal effects in random scattering media on superresolution, shower curtain effects, and backscattering enhancement*, Radio Science '07.

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• Endrei & Scarcelli, *Optical imaging through dynamic turbid media using the Fourier-domain shower-curtain effect, Optica* '16.

Random medium & and ε-scaling

- Important parameters: Wave-length $\lambda_0 = \frac{2\pi c_0}{\omega} = \frac{2\pi}{k}$; Beam Width r_0;

Propagation distance L;

Additionally random case: Medium coherence length ℓ & fluctuation strength σ.

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• Medium fluctuations:

$$c^{-2}(z,\mathbf{x}) = c_o^{-2} \begin{cases} 1 + \sigma\mu\left(\frac{z}{\ell}, \frac{\mathbf{x}}{\ell}\right) & \text{if} \quad z \in (z_a, z_b), \\ 1 & \text{else} \end{cases}$$

with μ zero-mean, stationary random field, strongly mixing in z.

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• Governing statistics: The *Lateral Spectrum* (in non-dimensionalized coordinates zooming in on the beam):

$$C(\tilde{\mathbf{x}}) = \int_{-\infty}^{\infty}\mathbb{E}\left[\mu(0,\mathbf{0})\mu(\tilde{z},\tilde{\mathbf{x}})\right]d\tilde{z}, \quad C(\mathbf{0})<\infty.$$

• The isotropic paraxial scaling used here $\lambda_0 \ll \ell, r_0 \ll L$:

$$\frac{\lambda_o}{L} = O(\varepsilon^2); \quad \frac{r_0}{L}\sim\frac{\ell}{L} = O(\varepsilon); \quad \sigma = \varepsilon^{3/2}.$$

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• Borcea, Garnier & S. *Paraxial wave propagation in random media with long-range correlations*, SIAP '23.

Time harmonic form of scalar wave equation:

$$(\partial_z^2 + \Delta_\perp)\hat{u} + k^2 \hat{u} = 0,$$

boundary/radiation conditions. $k = \omega / c_0$ is the wave number.

• Slowly-varying envelope around a plane wave going in the z direction

$$\hat{u}(\omega, z, \mathbf{x}) = e^{ikz} \hat{a}(\omega, z, \mathbf{x})$$

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$\rightarrow$ Diffractive effects (spreading/bending of beam) in homogeneous medium of order one ($\lambda_0 \ll \ell, r_0 \ll L$):

$$\underbrace{\partial_z^2 \hat{a}}_{\sim \frac{1}{L^2}} \quad + \quad \underbrace{2ik\partial_z \hat{a}}_{\sim \frac{1}{\lambda_0 L}} \quad + \quad \underbrace{\Delta_\perp \hat{a}}_{\sim \frac{1}{r_0^2}} \quad = 0$$

$$\sim \frac{1}{L^2} \quad \ll \quad \sim \frac{1}{\lambda_0 L} \quad \approx \quad \sim \frac{1}{r_0^2}$$

Itô-Schrödinger and damping of mean field

• Wavefield described in distribution in regime of small ε by:

$$\to d_z\hat{a} = \frac{1}{2ik}\Delta_\perp \hat{a} - \frac{k^2 C(\mathbf{0})}{8}\hat{a} + \frac{ik}{2}\hat{a}\, dW_z, \quad \hat{u}(\omega, z, \mathbf{x}) = e^{ikz}\hat{a}(\omega, z, \mathbf{x})$$

gives (with Brownian field W having lateral spectrum C):

$$\partial_z \mathbb{E}[\hat{a}] = \frac{1}{2ik}\Delta_\perp \mathbb{E}[\hat{a}] - \frac{k^2 C(\mathbf{0})}{8}\mathbb{E}[\hat{a}].$$

Then

$$\mathbb{E}[\hat{a}(\omega, z, \mathbf{x})] = \hat{a}_0(\omega, z, \mathbf{x}) \exp\left(-\frac{z}{\ell_{\rm sca}}\right),$$

for $\hat{a}_0$ solution in homogeneous medium and we defined the scattering mean free path:

$$\ell_{\rm sca} = \frac{8}{k^2 C(\mathbf{0})} = \frac{2}{\pi^2}\frac{\lambda_0^2}{C(\mathbf{0})} \sim L$$

• Garnier & S. *Coupled paraxial wave equations in random media in the white-noise regime*, Annals of Applied Prob '09.

Warm-up: Field Observations and Matched Field Imaging

• Compute correlation between the observed field at $z = L$ and the synthetic field in homogeneous medium generated by a point source at $\mathbf{x}$

Set-up: L, c_o known; infinite aperture $D = \infty$; $a \mapsto \hat{u}$,
$\Delta z = z_b - z_a$ width of complex section:

$$\mathcal{U}(\mathbf{x}) = \frac{ik_o}{2\pi L} \int_{\mathbb{R}^2} \hat{u}(\mathbf{y}, L) \exp\left(- \frac{ik_o |\mathbf{x} - \mathbf{y}|^2}{2L} \right) \mathrm{d}\mathbf{y}.$$

$$\mathbb{E}\big[\mathcal{U}(\mathbf{x})\big] = f(\mathbf{x}) \exp\left(- \frac{k_o^2 C(\mathbf{0}) \Delta z}{8} \right) = \underbrace{f(\mathbf{x})}_{\text{'source'}} \exp\left(- \frac{\Delta z}{\ell_{\mathrm{sca}}} \right)$$

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- No shower curtain effect for resolution.
- Poor imaging with strong clutter when $\Delta z \gtrapprox \ell_{\rm sca}$.
- In case with finite detector aperture D image additionally blurred with Gaussian kernel with width $\lambda_o L/D$ (the Rayleigh resolution formula).

Remark: Wigner Transform and Second Moment

• The *mean* Wigner transform is defined by

$$W_{\mathrm{m}}(\mathbf{x},\xi,z) := \int_{\mathbb{R}^2} \exp\left(-i\xi\cdot\mathbf{q}\right)\mathbb{E}\left[\hat{u}\left(\mathbf{x}+\frac{\mathbf{q}}{2},z\right)\overline{\hat{u}}\left(\mathbf{x}-\frac{\mathbf{q}}{2},z\right)\right]\mathrm{d}\mathbf{q},$$

and satisfies a (closed) transport equation and represents an angularly resolved wave energy density.

• The covariance of the image random field can be expressed in terms of the mean Wigner transform:

$$\mathbb{E}\left[\mathcal{U}(\mathbf{x}+\frac{\rho}{2})\overline{\mathcal{U}(\mathbf{x}-\frac{\rho}{2})}\right] \sim \int_{\mathbb{R}^2} W_{\mathrm{m}}\left(\mathbf{r},\frac{k_o}{z_1}(\mathbf{r}-\mathbf{x}),z_1\right)\exp\left(\frac{ik_o}{z_1}(\mathbf{r}-\mathbf{x})\cdot\rho\right)\mathrm{d}\mathbf{r}.$$

• Assume smooth medium fluctuations: $C(\mathbf{x}) = C(\mathbf{0}) - \frac{|\mathbf{x}|^2}{\ell_{\rm par}} + o(|\mathbf{x}|^2)$, for the paraxial distance $\ell_{\rm par}$ the range of validity of the paraxial approximation. Then in strongly heterogeneous case $\Delta z \gg \ell_{\rm sca}$, we find using Wigner transform:

$$\mathrm{Var}(\mathcal{U}(\mathbf{x})) = \int_{\mathbb{R}^2} |f(\mathbf{r})|^2 \mathcal{Q}(\mathbf{x}-\mathbf{r}) d\mathbf{r},$$

$$\mathcal{Q}(\mathbf{r}) = (2\pi\mathcal{R})^{-1} \exp\Big(-\frac{|\mathbf{r}|^2}{2\mathcal{R}^2}\Big),$$

Matched Field Image Fluctuations

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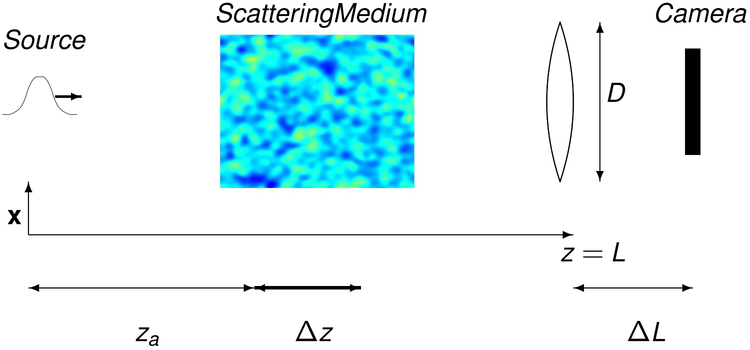
• The shower curtain spreading scale:

$$\mathcal{R} = \mathcal{R}(z_a, \Delta z, \ell_{\rm par}) = \sqrt{\frac{(z_a+\Delta z)^3 - z_a^3}{6\ell_{\rm par}}} \propto \begin{cases} \Delta z \sqrt{\frac{\Delta z}{\ell_{\rm par}}} & \text{for } z_a \ll \Delta z \\ z_a \sqrt{\frac{\Delta z}{\ell_{\rm par}}} & \text{for } z_a \gg \Delta z \end{cases},$$

Next: Optical Imaging



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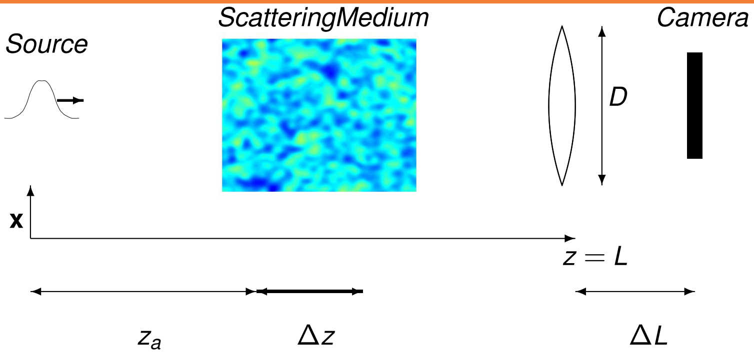

$\rightarrow$ The recorded intensity in camera plane is :

$$I(\mathbf{x}) \sim \left| \int_{\mathbb{R}^2} \hat{u}(\mathbf{y}, L)\mathcal{T}(\mathbf{y}) \exp\left(i \frac{k_o |\mathbf{x}-\mathbf{y}|^2}{2\Delta L} \right) d\mathbf{y} \right|^2 .$$

Transmission function of lens: $\mathcal{T}(\mathbf{y}) = \exp\left(-i \frac{k_o |\mathbf{y}|^2}{\mathcal{L}} - \frac{|\mathbf{y}|^2}{2D^2} \right)$.

Photodetector placed so that $\frac{1}{\mathcal{L}} = \frac{1}{L} + \frac{1}{\Delta L}$, $\mathcal{L} \sim$ focal length of lens.

Quantitative Description of Resolution

- Mean image in terms of the mean Wigner transform, $D = \infty$:

$$\mathbb{E}[I(\mathbf{x})] =\sim \int_{\mathbb{R}^2} W_{\rm m}\Big(\mathbf{r}, \frac{k_o}{\mathcal{L}}\mathbf{r} + \frac{k_o}{\Delta L}(\mathbf{x}-\mathbf{r}), L\Big)\mathrm{d}\mathbf{r} \sim I_{\rm m}\Big(-\mathbf{x}\frac{L}{\Delta L}\Big),$$

$$I_{\rm m}(\mathbf{x}) = \int_{\mathbb{R}^2} |f(\mathbf{r})|^2 \mathcal{H}(\mathbf{x}-\mathbf{r})\mathrm{d}\mathbf{r},$$

$$\mathcal{H}(\mathbf{y}) = \exp\left(-\frac{2\Delta z}{\ell_{\rm sca}}\right)\delta(\mathbf{y}) + \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \exp\left(i\zeta\cdot\mathbf{y}\right)$$

$$\times\left[\exp\left(\frac{k_o^2}{4}\int_{z_a}^{z_a+\Delta z} C\left(\zeta\frac{z}{k_o}\right) - C(\mathbf{0})dz\right) - \exp\left(-\frac{2\Delta z}{\ell_{\rm sca}}\right)\right]\mathrm{d}\zeta.$$

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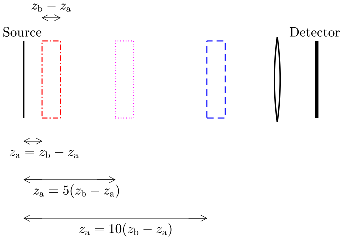
- In smooth strongly heterogeneous case with finite D we have:

$$\mathcal{H}(\mathbf{y}) \sim \exp\left(-\frac{|\mathbf{y}|^2}{2\left(\mathcal{R}^2 + \left(\frac{L}{k_o D}\right)^2\right)}\right), \quad \mathcal{R} \approx \begin{cases} \Delta z\sqrt{\frac{\Delta z}{\ell_{\rm par}}} & \text{for } z_a \ll \Delta z \\ z_a\sqrt{\frac{\Delta z}{\ell_{\rm par}}} & \text{for } z_a \gg \Delta z \end{cases}$$

Performance Example

Imaging with 3 locations of random section:



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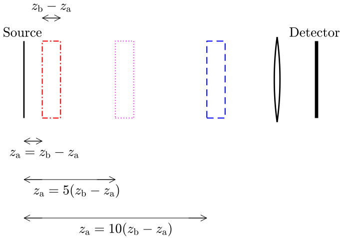

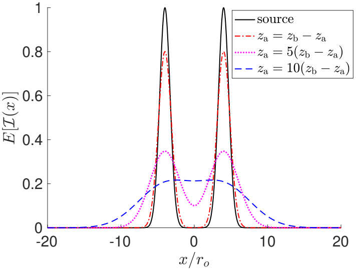

Figure: Left blurring of mean image increases when the random section is farther from the source, $f(x) = \exp(-(x-4r_o)^2/(2r_o^2)) + \exp(-(x+4r_o)^2/(2r_o^2))$ and $\mathcal{R}^2 = \frac{(z_b - z_a)^3}{6\ell_{\rm par}} = \left(\frac{r_o}{5}\right)^2$.

Quantitative Description of Stability

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$\to$ Need to understand the second-order moment of the imaging function

$$\mathbb{E}\left[I(\mathbf{x})^2\right] \sim \iint_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2 \mathrm{d}\mathbf{y}_1 \mathrm{d}\mathbf{y}_2$$
$$\times\, \mathbb{E}\left[\hat{u}(\mathbf{x}_1, L)\hat{u}(\mathbf{x}_2, L)\overline{\hat{u}(\mathbf{y}_1, L)\hat{u}(\mathbf{y}_2, L)}\right]$$
$$\times \exp\left(-\frac{ik_o}{2\mathcal{L}}\left[|\mathbf{x}_1|^2 - |\mathbf{y}_1|^2 + |\mathbf{x}_2|^2 - |\mathbf{y}_2|^2\right]\right)$$
$$\times \exp\left(\frac{ik_o}{2\Delta L}\left[|\mathbf{x}_1-\mathbf{x}|^2 - |\mathbf{y}_1-\mathbf{x}|^2 + |\mathbf{x}_2-\mathbf{x}|^2 - |\mathbf{y}_2-\mathbf{x}|^2\right]\right).$$

Closed expression for second moment of imaging function in scintillation regime ($\delta \ll 1$):

$$r_0 \to \frac{r_0}{\delta}, \quad C \to \delta C, \quad z \to \frac{z}{\delta}.$$

$\hookrightarrow$ Calculation reveals lack of statistical stability: $\mathrm{Var}(I(\mathbf{x})) = \mathbb{E}[I(\mathbf{x})]^2$!

Broadband to the Rescue

- Use a broadband source:

$$f(\mathbf{x},t) = g(t)\exp\Big(-\frac{|\mathbf{x}|^2}{2r_0^2}\Big) + c.c., \quad \hat{g}(\omega) = \frac{1}{\sqrt{B}}\hat{g}_0\Big(\frac{\omega-\omega_0}{B}\Big),$$

and consider the optical imaging function the spatially resolved total wave energy recorded by the photodetector in the camera plane ($D=\infty$):

$$I(\mathbf{x}) \sim \int_{\mathbb{R}} \mathrm{d}t \Big| \int_{\mathbb{R}} \mathrm{d}\omega \int_{\mathbb{R}^2} \mathrm{d}\mathbf{y}\, \hat{u}(\omega,\mathbf{y},L)\exp\Big(i\frac{k(\omega)|\mathbf{x}-\mathbf{y}|^2}{2\Delta L} - i\frac{k(\omega)|\mathbf{y}|^2}{2\mathcal{L}} - i\omega t\Big)\Big|^2.$$

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$\rightarrow$ Multifrequency fourth moment gives (strong & smooth case):

- Resolution as in one frequency case, ($B \ll \omega_0$) !
- $\dfrac{\mathbb{E}\left[\mathcal{I}(\mathbf{x})\right]^2}{\mathrm{Var}\left(\mathcal{I}(\mathbf{x})\right)} = O\left(\frac{B}{\Omega_{\mathrm{c}}}\right) \propto (\Delta z)^2.$

for the coherence frequency: $\Omega_{\mathrm{c}} = \left(\frac{c_o \ell_{\mathrm{par}}}{\Delta z^2}\right) = T_{\mathrm{r}}^{-1}\left(\frac{\ell_{\mathrm{par}}}{\Delta z}\right)$,

(frequency correlation band $\sim$ reciprocal passage time).

Garnier & S., *Shower curtain effect and source imaging*, IPI '24.
Derode, Tourin & Fink, *Random scattering of ultrsound; is TR self-averaging?* Phys Rev E '01.

Summary

- Have developed a theory for shower curtain effect in high frequency paraxial regime in a situation with a complex section.
- In a regime of relatively strong scattering anomalous spreading associated with a shower curtain effect.
- Quantitative description of resolution and stability.
- Multifrequency information important for statistical stability.

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- Quantitative description of resolution and stability.
- Multifrequency information important for statistical stability.
- In progress with Christophe Gomez; Shower curtain with rough surface:
  • Main modelling Quantity interface diffraction operator:

$$K^{\varepsilon}(\tau,\omega,\mathbf{q}) = \frac{\omega^2}{(2\pi)^2}\int e^{i\omega\mathbf{q}\cdot(\mathbf{x}'-\mathbf{x}_a)/(r_0/L)} e^{i\omega\tau V(\mathbf{x}'/(\ell_c/L))} d\mathbf{x}'.$$

Technical remarks: On Multifrequency Moment

We consider two frequencies ω_1, ω_2 and the fourth-order moment

$$M_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2, z) = \frac{\mathbb{E}\Big[\hat{u}(\omega_1, \mathbf{x}_1, z)\hat{u}(\omega_2, \mathbf{x}_2, z)\overline{\hat{u}(\omega_1, \mathbf{y}_1, z)\hat{u}(\omega_2, \mathbf{y}_2, z)}\Big]}{|\hat{g}(\omega_1)|^2 |\hat{g}(\omega_2)|^2}.$$

It satisfies

$$\frac{\partial M_2}{\partial z} = \frac{i}{2}\Big(\frac{1}{k_1}\Delta_{\mathbf{x}_1} + \frac{1}{k_2}\Delta_{\mathbf{x}_2} - \frac{1}{k_1}\Delta_{\mathbf{y}_1} - \frac{1}{k_2}\Delta_{\mathbf{y}_2}\Big) M_2$$
$$-U_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2)\mathbf{1}_{[z_a, z_a+\Delta z]}(z) M_2,$$

with the generalized potential

$$U_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2) = \frac{1}{4}\Big((k_1^2 + k_2^2)C(\mathbf{0}) - k_1^2 C(\mathbf{x}_1 - \mathbf{y}_1) - k_2^2 C(\mathbf{x}_2 - \mathbf{y}_2)$$
$$-k_1 k_2 C(\mathbf{x}_1 - \mathbf{y}_2) - k_1 k_2 C(\mathbf{x}_2 - \mathbf{y}_1) + k_1 k_2 C(\mathbf{x}_1 - \mathbf{x}_2) + k_1 k_2 C(\mathbf{y}_1 - \mathbf{y}_2)\Big)$$
$$= \frac{1}{8}\mathbb{E}\Big[\big(k_1(\tilde{\mu}(\mathbf{x}_1) - \tilde{\mu}(\mathbf{y}_1)) + k_2(\tilde{\mu}(\mathbf{x}_2) - \tilde{\mu}(\mathbf{y}_2))\big)^2\Big].$$

On Analysis I

- Consider the narrow band scintillation regime:

$$r_0 \to \frac{r_0}{\delta}, \quad C \to \delta C, \quad z \to \frac{z}{\delta}, \quad B \to \delta B.$$

- Introduce the special Fourier transform of the fourth-order moment :

$$\widetilde{M}_2^\delta(\xi_1, \xi_2, \zeta_1, \zeta_2, \frac{z}{\delta}) = \iint_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} M_2^\delta(\mathbf{q}_1, \mathbf{q}_2, \mathbf{r}_1, \mathbf{r}_2, \frac{z}{\delta})$$
$$\times e^{-i(\mathbf{q}_1 \cdot \xi_1 + \mathbf{r}_1 \cdot \zeta_1 + \mathbf{q}_2 \cdot \xi_2 + \mathbf{r}_2 \cdot \zeta_2} \mathrm{d}\mathbf{r}_1 \mathrm{d}\mathbf{r}_2 \mathrm{d}\mathbf{q}_1 \mathrm{d}\mathbf{q}_2 e^{\frac{iz}{k_o \delta}(\xi_2 \cdot \zeta_2 + \xi_1 \cdot \zeta_1)},$$

for $\mathbf{r}_j, \mathbf{q}_j$ Barycentric coordinates.

On Analysis II

For frequency parameterization: $\omega_1 = \omega_0 + \delta\omega + \delta\Omega, \qquad \omega_2 = \omega_0 + \delta\omega - \delta\Omega,$ in the scintillation regime the rescaled function $\widetilde{M}_2^\delta$ satisfies the equation with fast phases :

$$
\begin{aligned}
\frac{\partial \widetilde{M}_2^\delta}{\partial z} = \; & \frac{i\omega}{k_o\omega_0}\left(\xi_1\cdot\zeta_1 + \xi_2\cdot\zeta_2\right)\widetilde{M}_2^\delta + \frac{i\Omega}{k_o\omega_0}\left(\xi_1\cdot\zeta_2 + \xi_2\cdot\zeta_1\right)\widetilde{M}_2^\delta \\
& + \frac{k_o^2}{4(2\pi)^2}\mathbf{1}_{[z_a, z_a+\Delta z]}(z)\int_{\mathbb{R}^2}\hat{C}(\kappa)\Big[-2\widetilde{M}_2^\delta(\xi_1,\xi_2,\zeta_1,\zeta_2) \\
& + \widetilde{M}_2^\delta(\xi_1-\kappa,\xi_2-\kappa,\zeta_1,\zeta_2)e^{i\frac{z}{\delta k_o}\kappa\cdot(\zeta_2+\zeta_1)} \\
& + \widetilde{M}_2^\delta(\xi_1-\kappa,\xi_2,\zeta_1,\zeta_2-\kappa)e^{i\frac{z}{\delta k_o}\kappa\cdot(\xi_2+\zeta_1)} \\
& + \widetilde{M}_2^\delta(\xi_1+\kappa,\xi_2-\kappa,\zeta_1,\zeta_2)e^{i\frac{z}{\delta k_o}\kappa\cdot(\zeta_2-\zeta_1)} \\
& + \widetilde{M}_2^\delta(\xi_1+\kappa,\xi_2,\zeta_1,\zeta_2-\kappa)e^{i\frac{z}{\delta k_o}\kappa\cdot(\xi_2-\zeta_1)} \\
& - \widetilde{M}_2^\delta(\xi_1,\xi_2-\kappa,\zeta_1,\zeta_2-\kappa)e^{i\frac{z}{\delta k_o}(\kappa\cdot(\zeta_2+\xi_2)-|\kappa|^2)} \\
& - \widetilde{M}_2^\delta(\xi_1,\xi_2-\kappa,\zeta_1,\zeta_2+\kappa)e^{i\frac{z}{\delta k_o}(\kappa\cdot(\zeta_2-\xi_2)+|\kappa|^2)}\Big]d\kappa,
\end{aligned}
$$

Proposition

$$\widetilde{M}_2^\delta(\xi_1,\xi_2,\zeta_1,\zeta_2,\frac{z}{\delta}) = \mathcal{V}(K,A)_z + R_2^\delta(z,\xi_1,\xi_2,\zeta_1,\zeta_2),$$

$$\text{with } K(z) = (2\pi)^4 \exp\Big(-\frac{k_o^2}{4} C(\mathbf{0}) \min(\Delta z,(z-z_a)_+)\Big),$$

and where the function $(z,\xi) \mapsto A(z,\xi,\zeta,\Omega)$ *is the solution of*

$$\begin{aligned}\partial_z A \;=\; & \frac{i\Omega}{k_o\omega_0}|\xi|^2 A + \frac{k_o^2}{4(2\pi)^2}\mathbf{1}_{[z_a,z_a+\Delta z]}(z)\int_{\mathbb{R}^2}\hat{C}(\kappa)\big[A(\xi-\kappa)e^{\frac{iz}{k_o}\kappa\cdot\zeta} - A(\xi)\big]d\kappa \\ & + \frac{k_o^2}{4(2\pi)^2}K(z)\mathbf{1}_{[z_a,z_a+\Delta z]}(z)\hat{C}(\xi)e^{\frac{iz}{k_o}\xi\cdot\zeta},\end{aligned}$$

starting from $A(z=0,\xi,\zeta,\Omega)=0$, *and the function* R_2^δ *satisfies*

$$\sup_{z\in[0,z_1]} \|R_2^\delta(z,\cdot,\cdot,\cdot,\cdot)\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2)} \xrightarrow{\delta\to 0} 0.$$

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Remark: Calculus of the Gaussian

First wave moments:

$$\begin{aligned}\mu_1(z,\mathbf{x}) &= \mathbb{E}[u(z,\mathbf{x})], \quad \mu_2(z,\mathbf{x},\mathbf{y}) = \mathbb{E}[u(z,\mathbf{x})\overline{u(z,\mathbf{y})}],\\ \tilde{\mu}_2(z,\mathbf{x},\mathbf{y}) &= \mu_2(z,\mathbf{x},\mathbf{y}) - \mu_1(z,\mathbf{x})\mu_1(z,\mathbf{y})\end{aligned}$$

$\rightarrow$ Assuming complex circularly symmetric Gaussian process property:

$$\begin{aligned}&\mathbb{E}[u(z,\mathbf{x}_1)u(z,\mathbf{x}_2)\overline{u(z,\mathbf{y}_1)u(z,\mathbf{y}_2)}] = \mu_4^G(z,\mathbf{x}_1,\mathbf{x}_2,\mathbf{y}_1,\mathbf{y}_2)\\ &= \mu_1(z,\mathbf{x}_1)\mu_1(z,\mathbf{x}_2)\mu_1(z,\mathbf{y}_1)\mu_1(z,\mathbf{y}_2)\\ &+\mu_1(z,\mathbf{x}_1)\mu_1(z,\mathbf{y}_1)\tilde{\mu}_2(z,\mathbf{x}_2,\mathbf{y}_2) + \mu_1(z,\mathbf{x}_2)\mu_1(z,\mathbf{y}_1)\tilde{\mu}_2(z,\mathbf{x}_1,\mathbf{y}_2)\\ &+\mu_1(z,\mathbf{x}_1)\mu_1(z,\mathbf{y}_2)\tilde{\mu}_2(z,\mathbf{x}_2,\mathbf{y}_1) + \mu_1(z,\mathbf{x}_2)\mu_1(z,\mathbf{y}_2)\tilde{\mu}_2(z,\mathbf{x}_1,\mathbf{y}_1)\\ &+\tilde{\mu}_2(z,\mathbf{x}_1,\mathbf{y}_1)\tilde{\mu}_2(z,\mathbf{x}_2,\mathbf{y}_2) + \tilde{\mu}_2(z,\mathbf{x}_1,\mathbf{y}_2)\tilde{\mu}_2(z,\mathbf{x}_2,\mathbf{y}_1)\end{aligned}$$

Quasi-Gaussianity

$$
\begin{aligned}
\mathcal{V}(K,A)_z &= K(z)^2\phi_{r_0}^{\delta}(\xi_1)\phi_{r_0}^{\delta}(\xi_2)\phi_{r_0}^{\delta}(\zeta_1)\phi_{r_0}^{\delta}(\zeta_2) \\
&+\frac{K(z)}{2}\phi_{r_0}^{\delta}\big(\frac{\xi_1-\xi_2}{\sqrt{2}}\big)\phi_{r_0}^{\delta}(\zeta_1)\phi_{r_0}^{\delta}(\zeta_2)A\big(z,\frac{\xi_2+\xi_1}{2},\frac{\zeta_2+\zeta_1}{\delta},0\big) \\
&+\frac{K(z)}{2}\phi_{r_0}^{\delta}\big(\frac{\xi_1+\xi_2}{\sqrt{2}}\big)\phi_{r_0}^{\delta}(\zeta_1)\phi_{r_0}^{\delta}(\zeta_2)A\big(z,\frac{\xi_2-\xi_1}{2},\frac{\zeta_2-\zeta_1}{\delta},0\big) \\
&+\frac{K(z)}{2}\phi_{r_0}^{\delta}\big(\frac{\xi_1-\zeta_2}{\sqrt{2}}\big)\phi_{r_0}^{\delta}(\zeta_1)\phi_{r_0}^{\delta}(\xi_2)A\big(z,\frac{\zeta_2+\xi_1}{2},\frac{\xi_2+\zeta_1}{\delta},\Omega\big) \\
&+\frac{K(z)}{2}\phi_{r_0}^{\delta}\big(\frac{\xi_1+\zeta_2}{\sqrt{2}}\big)\phi_{r_0}^{\delta}(\zeta_1)\phi_{r_0}^{\delta}(\xi_2)A\big(z,\frac{\zeta_2-\xi_1}{2},\frac{\xi_2-\zeta_1}{\delta},-\Omega\big) \\
&+\frac{1}{4}\phi_{r_0}^{\delta}(\zeta_1)\phi_{r_0}^{\delta}(\zeta_2)A\big(z,\frac{\xi_2+\xi_1}{2},\frac{\zeta_2+\zeta_1}{\delta},0\big)\times A\big(z,\frac{\xi_2-\xi_1}{2},\frac{\zeta_2-\zeta_1}{\delta},0\big) \\
&+\frac{1}{4}\phi_{r_0}^{\delta}(\zeta_1)\phi_{r_0}^{\delta}(\xi_2)A\big(z,\frac{\zeta_2+\xi_1}{2},\frac{\xi_2+\zeta_1}{\delta},\Omega\big)\times A\big(z,\frac{\zeta_2-\xi_1}{2},\frac{\xi_2-\zeta_1}{\delta},-\Omega\big)
\end{aligned}
$$

On Focussing Classic Time-reversal

Coherence/speckle frequency:

$$\Omega_{\rm c} := T^{-1}\,\frac{\ell_{\rm par}}{L},$$

for $T = L/c_o$ is the travel time over the distance L from the source to the TRM for a background wave speed c_o and $\ell_{\rm par}$ is the paraxial distance.

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• In scintillation scaling:

$$\mathrm{SNR} \approx N \max\left\{\frac{B}{\Omega_c}, 1\right\} \quad \text{for} \quad N := \left(\frac{R_0}{\rho_0}\right)^2,$$

with R_0 being the size of the TRM and $\rho_0 = O(R_0)$ the size of the elements.

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• Classic result for refocussing resolution ($\ell_\mathrm{par} > L$):

$$\mathcal{R} \approx \lambda_o \sqrt{\frac{\ell_\mathrm{par}}{L}}.$$

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Shower Curtain, Time Reversal and Reciprocity

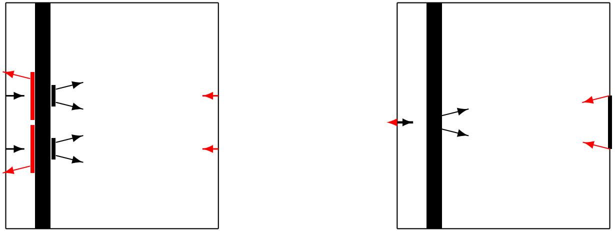

Figure: Left: Shower Curtain Right: Time Reversal

Physical intuition: *Medium* $\sim$ *low pass filter*.

Various

- Important parameters: paraxial distance, coherence frequency, scattering mean free path
- Additional effects of medium roughness, partly coherent source
- Application in & active configurations & speckle imaging & virtual aperture configurations